# REAL ALGEBRAIC GEOMETRY LECTURE NOTES (09: 17/11/09)

SALMA KUHLMANN

Contents



## 1. Basic version of Tarski-Seidenberg

**Basic version**: Let $(R, \leqslant)$ be a real closed field. We are interested in a system of equations and inequalities (*Gleichungen und Ungleichungen*) for $\underline{X} = (X_1, \dots, X_n)$ of the form

$$S(\underline{X}) := \begin{cases} f_1(\underline{X}) \lhd_1 0 \\ \quad \vdots \\ f_k(\underline{X}) \lhd_k 0 \end{cases}$$

where $\forall\, i = 1, \dots, k \quad \lhd_i \in \{\geqslant, >, =, \neq\}$ and $f_i(\underline{X}) \in \mathbb{Q}[\underline{X}]$ or $f_i(\underline{X}) \in R[\underline{X}]$. We say that $S(\underline{X})$ is a system of polynomial equalities and inequalities with coefficients in $\mathbb{Q}$ (or with coefficients in $R$) in $n$ variables.

**Theorem 1.1.** *(Tarski-Seidenberg Theorem: Basic Version) Let $S(\underline{T}; \underline{X})$ be a system with coefficients in $\mathbb{Q}$ in $m+n$ variables, with $\underline{T} = (T_1, \dots, T_m)$ and $\underline{X} = (X_1, \dots, X_n)$. Then there exist $S_1(\underline{T}), \dots, S_l(\underline{T})$ systems in $m$ variables and coefficients in $\mathbb{Q}$ such that:*

*for every real closed field $R$ and every $\underline{t} = (t_1, \dots, t_m) \in R^m$ the system $S(\underline{t}; \underline{X})$ of polynomial equalities and inequalities in $n$ variables and coefficients in $R$ obtained by substituting $T_i$ with $t_i$ in $S(\underline{T}, \underline{X})$ for every $i = 1, \dots, m$, has a solution $\underline{x} = (x_1, \dots, x_n) \in R^n$ if and only if $\underline{t} = (t_1, \dots, t_m) \in R^m$ is a solution in $R$ for one of the systems $S_1(\underline{T}), \dots, S_l(\underline{T})$.*

**Example 1.2.** Let $m = 3$ and $n = 1$, so $\underline{T} = (T_1, T_2, T_3)$ and $\underline{X} = X$, and

$$S(\underline{T}, \underline{X}) := \Big\{ T_1 X^2 + T_2 X + T_3 = 0$$

Let $R$ be a real closed field and $(t_1, t_2, t_3) \in R^3$. Then $S(\underline{t}; X)$ has a solution $X$ in $R$ if and only if

$$\begin{array}{ccccc} (t_1 \neq 0 \ \wedge \ t_2^2 - 4t_1t_3 \geqslant 0) & \vee & (t_1 = 0 \ \wedge \ t_2 \neq 0) & \vee & (t_1 = t_2 = t_3 = 0) \\ | & & | & & | \\ S_1(T_1, T_2, T_3) & & S_2(T_1, T_2, T_3) & & S_3(T_1, T_2, T_3) \end{array}$$

**Concise version**:

$$\forall \underline{T} \ [ \ (\exists \underline{X} : S(\underline{T}; \underline{X})) \ \Leftrightarrow \ (\bigvee_{i=1}^{l} S_i(\underline{T})) \ ].$$

**Remark 1.3.** The proof is by induction on $n$.
The case $n = 1$ is the heart of the proof and we will show it later.
For now, let us just convince ourselves that the induction step is straightforward.

Assume $n > 1$, so

$$S(\underline{T}, X_1, \dots, X_n) = S(\underline{T}, X_1, \dots, X_{n-1}; X_n).$$

By case $n = 1$ we have finitely many systems $S_1(\underline{T}, X_1, \dots, X_{n-1}), \dots, S_l(\underline{T}, X_1, \dots, X_{n-1})$ such that

for any real closed field $R$ and any $(t_1, \dots, t_m, x_1, \dots, x_{n-1}) \in R^{m+n-1}$ we have

$$\exists X_n : \ S(t_1, \dots, t_m, x_1, \dots, x_{n-1}; X_n) \ \longleftrightarrow \ \bigvee_{i=1}^{l} S_i(t_1, \dots, t_m, x_1, \dots, x_{n-1}).$$

By induction hypothesis on $n$:

for every fixed $i$, $1 \leqslant i \leqslant l$, $\exists$ systems $S_{ij}(\underline{T})$, $j = 1, \dots, l_i$ such that: for each real closed field $R$ and each $\underline{t} \in R^m$ the system

$$S_i(\underline{t}; X_1, \dots, X_{n-1})$$

has a solution $(x_1, \dots, x_{n-1}) \in R^{n-1}$ if and only if $\underline{t}$ is a solution for one of the systems $S_{ij}(\underline{T})$; $j = 1, \dots, l_i$.
Therefore for any real closed field $R$ and any $\underline{t} \in R^m$

$$S(\underline{t}; X_1, \dots, X_n) \text{ has a solution } \underline{x} \in R^n \text{ if and only if}$$

$$\underline{t} \text{ is a solution to one of the systems } \Big\{ S_{ij}(\underline{T}); \ i = 1, \dots, l, \ j = 1, \dots, l_i$$

## 2. Tarski Transfer Principle I

**Theorem 2.1.** *Let $S(\underline{T}, \underline{X})$ be a system with coefficients in $\mathbb{Q}$ in $m + n$ variables. Let $(K, \leqslant)$ be an ordered field. Let $R_1, R_2$ be two real closed extensions of $(K, \leqslant)$. Then for every $\underline{t} \in K^m$, the system $S(\underline{t}, \underline{X})$ has a solution $\underline{x} \in R_1^n$ if and only if it has a solution $\underline{x} \in R_2^n$.*

*Proof.* Let $\underline{t} \in K^m \subseteq R_1^m \cap R_2^m$. Then there are systems $S_i(\underline{T})$ $(i = 1, \dots, l)$ with coefficients in $\mathbb{Q}$ and variables $T_1, \dots, T_m$ such that

$$\exists\, \underline{x} \in R_1 : S(\underline{t}, \underline{x}) \;\longleftrightarrow\; \underline{t} \;\text{ satisfies } \bigvee_{i=1}^{l} S_i(\underline{T}) \;\longleftrightarrow\; \exists\, \underline{x} \in R_2 : S(\underline{t}, \underline{x}).$$

□

## 3. Tarski Transfer Principle II

**Theorem 3.1.** *Let $(K, \leqslant)$ be an ordered field, $R_1, R_2$ two real closed extensions of $(K, \leqslant)$. Then a system of polynomial equations and inequalities of the form*

$$S(\underline{X}) := \begin{cases} f_1(\underline{X}) \lhd_1 0 \\ \quad\vdots \\ f_k(\underline{X}) \lhd_k 0 \end{cases}$$

*where* $\forall\, i = 1, \dots, k \quad \lhd_i \in \{\geqslant, >, =, \neq\}$ *and* $f_i(\underline{X}) \in K[X_1, \dots, X_n]$,

*has a solution* $\underline{x} \in R_1^n \iff$ *it has a solution* $\underline{x} \in R_2^n$.

*Proof.* Let $t_1, \dots, t_m$ be the coefficients of the polynomials $f_1, \dots, f_k$, listed in some fixed order. Replacing the coefficients $t_1, \dots, t_m$ by variables $T_1, \dots, T_m$ yields a system $\sigma(\underline{T}, \underline{X})$ in $m + n$ variables with coefficients in $\mathbb{Q}$ (in fact in $\mathbb{Z}$) for which

$$\sigma(t_1, \dots, t_m, \underline{X}) = S(\underline{X}).$$

Now we can apply Tarski Transfer I. □

## 4. Tarski Transfer Principle III

**Theorem 4.1.** *Suppose that $R \subseteq R_1$ are real closed fields. Then a system of polynomial equations and inequalities with coefficients in $R$*

$$S(\underline{X}) := \begin{cases} f_1(\underline{X}) \lhd_1 0 \\ \quad\vdots \\ f_k(\underline{X}) \lhd_k 0 \end{cases}$$

*where* $\forall\, i = 1, \dots, k \quad \lhd_i \in \{\geqslant, >, =, \neq\}$ *and* $f_i(\underline{X}) \in R[X_1, \dots, X_n]$

*has a solution* $\underline{x} \in R_1^n \iff$ *it has a solution* $\underline{x} \in R^n$.

*Proof.* Apply Tarski Transfer II with $K = R_2 = R$. □

## 5. Tarski Transfer Principle IV

**Theorem 5.1.** *Let $R$ be a real closed field and $(F, \leqslant)$ an ordered field extension of $R$. Then a system of polynomial equations and inequalities of the form*

$$S(\underline{X}) := \begin{cases} f_1(\underline{X}) \lhd_1 0 \\ \quad\vdots \\ f_k(\underline{X}) \lhd_k 0 \end{cases}$$

*where* $\forall\, i = 1, \ldots, k \quad \lhd_i \in \{\geqslant, >, =, \neq\}$ *and* $f_i(\underline{X}) \in R[X_1, \ldots, X_n]$

*has a solution* $\underline{x} \in F^n \iff$ *it has a solution* $\underline{x} \in R^n$.

*Proof.* Let $R_1$ be the real closure of the ordered field $(F, \leqslant)$ and apply Tarski Transfer III. □

## 6. Lang's Homomorphism Theorem

**Corollary 6.1.** *Suppose $R$ and $R_1$ are real closed fields, $R \subseteq R_1$. Then a system of polynomial equations of the form*

$$S(\underline{X}) := \begin{cases} f_1(\underline{X}) = 0 \\ \quad\vdots \\ f_k(\underline{X}) = 0 \end{cases} \qquad f_i(\underline{x}) \in R[X_1, \ldots, X_n]$$

*has a solution $\underline{x} \in R_1^n$ if and only if it has a solution $\underline{x} \in R^n$.*

*Proof.* Apply Tarski Transfer III. □

The previous Corollary is equivalent to the following:

**Theorem 6.2.** *(Homomorphism Theorem I). Let $R$ and $R_1$ be real closed fields, $R \subseteq R_1$. For any ideal $I \subseteq R[\underline{X}]$, if there exists an $R$-algebra homomorphism*

$$\varphi \colon R[\underline{X}]/I \longrightarrow R_1$$

*then there exists an $R$-algebra homomorphism*

$$\psi \colon R[\underline{X}]/I \longrightarrow R.$$

*Proof.* By Hilbert's Basis Theorem, $I$ is finitely generated, say $I = \langle f_1, \ldots, f_k \rangle$, with $f_1, \ldots, f_k \in R[\underline{X}]$. Consider the system

$$S(\underline{X}) := \begin{cases} f_1(\underline{X}) = 0 \\ \quad\vdots \\ f_k(\underline{X}) = 0 \end{cases}$$

**Claim**. There is a bijection

$$\{\underline{x} \in R_1^n \text{ solution to } S(\underline{X})\} \longleftrightarrow \{\varphi \colon R[\underline{X}]/I \to R_1 \;\; R\text{-algebra homomorphism}\}$$

*Proof of the claim*:

Let $\underline{x} \in R_1^n$ be a solution to $S(\underline{X})$; then the evaluation homomorphism

$$\begin{array}{rccc} \varphi\colon & R[\underline{X}]/I & \longrightarrow & R_1 \\ & f+I & \mapsto & f(\underline{x}) \end{array}$$

is well-defined and is an $R$-algebra homomorphism.

Conversely: assume that

$$\varphi\colon R[\underline{X}]/I \longrightarrow R_1$$

is an $R$-algebra homomorphism. Then for $\underline{e} = (e_1, \ldots, e_n)$ and $f = \sum \underline{a}_e \underline{X}^e = \sum a_{e_1 \ldots e_n} X_1^{e_1} \ldots X_n^{e_n} \in R[\underline{X}]$,

$$\varphi(f+I) = \sum \underline{a}_e \varphi(X_1+I)^{e_1} \cdots \varphi(X_n+I)^{e_n} = f(\varphi(X_1+I), \ldots, \varphi(X_n+I)).$$

In other words set $(x_1, \ldots, x_n) \in R_1^n$ to be defined by $x_1 := \varphi(X_1 + I), \ldots, x_n := \varphi(X_n + I)$, then $(x_1, \ldots, x_n)$ is a solution to $S(\underline{X})$ and the $R$-algebra homomorphism $\varphi$ is indeed given by point evaluation at $\underline{x} = (x_1, \ldots, x_n) \in R_1^n$.

Now apply Corollary 6.1.