REAL ALGEBRAIC GEOMETRY LECTURE NOTES (29: 09/02/10)

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Contents

1. Hardy fields

Definition 1.1. (Hardy field) Consider the set of all real valued functions defined on positive half lines:

$$\mathcal{F} := \{f \mid f\colon [a,\infty) \to \mathbb{R} \text{ or } f\colon (a,\infty) \to \mathbb{R},\ a \in \mathbb{R}\}.$$

For every $f, g \in \mathcal{F}$ we define

$$f \sim g \;\Leftrightarrow\; \exists\, N \in \mathbb{N} \text{ s.t. } f(x) = g(x)\ \forall\, x \geqslant N.$$

When $f \sim g$ we say that f and g **have the same germ at** ∞.
We identify $f \in \mathcal{F}$ with its germ $[f]$.

We denote by $\mathcal{G}$ the set of all germs. Note that $\mathcal{G}$ is a commutative ring with 1 by:

$$[f] + [g] := [f + g]$$
$$[f] \cdot [g] := [f \cdot g]$$

A subring H of $\mathcal{G}$ is a **Hardy field** if it is a field with respect to the operations above and it is closed under differentiation, i.e.

$$f \in H \;\Rightarrow\; f' \in H.$$

Remark 1.2. (defininig a total order on a Hardy field). Let H be a Hardy field and $f \in H$, $f \neq 0$.

Since $1/f \in H$, $f(x) \neq 0$ ultimately. Moreover since $f' \in H$, f is ultimately differentiable and thus ultimately continuous.

It follows that $\mathrm{sign}(f)$ is constant ultimately (i.e. f is strictly positive on some interval (N,∞) or f is strictly negative on some interval (N,∞)).

This key property allows us to define a total order on H:

Definition 1.3. Let H be a Hardy field. For every f, g we define

$$f > g \;\Leftrightarrow\; f - g \;\text{ is ultimately positive.}$$

Lemma 1.4. $>$ *above is an ordering on* H*.*

Examples 1.5.

(1) $\mathbb{Q}$ and $\mathbb{R}$ are Hardy fields consisting of just constant germs. They are archimedean Hardy fields.

(2) Let x denote the germ of the identity function. Then $\mathrm{x} > \mathbb{R}$ and $\mathbb{R}(\mathrm{x})$ is a non-archimedean Hardy field.

Lemma 1.6. *(Monotonicity) Let* H *be a Hardy field and* $f \in H$*,* $f' \neq 0$*. Since* f' *is ultimately positive or negative, it follows that* f *is ultimately increasing or decreasing. Therefore*

$$\exists \lim_{x \to \infty} f(x) \in \mathbb{R} \cup \{-\infty, +\infty\}.$$

2. The natural valuation of a Hardy field

Definition 2.1. (Valuation on H). Let H be a Hardy field. Define for $f, g \neq 0$

$$f \sim g \;\Leftrightarrow\; \lim_{x \to \infty} \frac{f(x)}{g(x)} = r \in \mathbb{R} \setminus \{0\}.$$

This is an equivalence relation. Denote the equivalence class of f by $v(f)$. Define

$$v(f) + v(g) := v(fg),$$

and

$$v(f) > v(g) \;\Leftrightarrow\; \lim_{x \to \infty} \frac{f(x)}{g(x)} = 0.$$

Lemma 2.2. *The map*

$$\begin{array}{rcl} H & \longrightarrow & H/\sim \cup \{\infty\} \\ 0 \neq f & \mapsto & v(f) \\ 0 & \mapsto & \infty \end{array}$$

is a valuation and it is equivalent to the natural valuation.

Remark 2.3.

$$R_v = \{f : \lim_{x \to \infty} f(x) \in \mathbb{R}\}.$$

$$I_v = \{f : \lim_{x \to \infty} f(x) = 0\}.$$

$$\mathcal{U}_v = \{f : \lim_{x \to \infty} f(x) \in \mathbb{R} \setminus \{0\}\}.$$