Prof. Dr. Salma Kuhlmann Dr. Itay Kaplan

MODEL THEORY – EXERCISE 1

To be submitted on Wednesday 20.04.2011 by 14:00.

Definition.

Suppose L is a signature, M, N are L-structures.
We always assume that L contains a special binary relation symbol $\approx$, which will always be interpreted as equality: For any structure M, $a, b \in M$,

$$a \approx^M b \Leftrightarrow a = b.$$

(1) A *homomorphism* $f : M \to N$ is a function such that
  (a) For any n-ary relation symbol R,
  $(a_1, \ldots, a_n) \in R^M \Rightarrow (f(a_1), \ldots, f(a_n)) \in R^N$.
  (b) For any n-ary function symbol F,
  $F^M(a_1, \ldots, a_n) = b \Rightarrow F^N(f(a_1), \ldots, f(a_n)) = f(b)$.
  (c) For any constant c, $f\left(c^M\right) = c^N$.
(2) An *embedding* $f : M \to N$ is a homomorphism $f : M \to N$ such that in (a) above, $\Rightarrow$ is replaced by $\Leftrightarrow$.
(3) A homomorphism is called an *isomorphism* if it is an embedding and it is onto.
(4) A homomorphism $f : M \to M$ is called an *automorphism* if it is an isomorphism from M onto M.
(5) Denote $M \cong N$ when there exists an isomorphism $f : M \to N$.
(6) A group $(G, +, <)$ is an *ordered abelian group* if $(G, +)$ is an abelian group, $(G, <)$ is a linear ordering and $a < b \Rightarrow a + c < b + c$ for all $a, b, c \in G$.

Question 1.

Let A, B, C be structures for a signature L.

(1) Show that embeddings are injective (one to one).
(2) Show that if $f : A \to B$ and $g : B \to C$ are homomorphisms then $g \circ f : A \to C$ is a homomorphism.
(3) Show that $f : A \to B$ is an isomorphism iff f is a homomorphism and there exists a homomorphism $g : B \to A$ such that $f \circ g = \mathrm{id}_B$, $g \circ f = \mathrm{id}_A$.
(4) Show that $\cong$ is an equivalence relation between L-structures.

Question 2.

(1) Let $L = \{P\}$ where P is a predicate (a unary or a 1-place relation symbol). Find an example of two L-structures A, B such that there exists an injective homomorphism from A onto B, but they are not isomorphic.

(2) Let L be the signature of groups, $L = \{+\}$ where $+$ is a 2-place (binary) function symbol (but you may write $a + b$ instead of $+(a, b)$). Let M, N be abelian groups. Show that a group homomorphism $h : M \to N$ is exactly a homomorphism of structures.

(3) Let $\mathsf{L} = \{+, <\}$ where $<$ is a binary relation symbol (but you may write $a < b$ instead of $< (a, b)$). Let M, N be ordered abelian groups. Show that if $f : M \to N$ is an injective homomorphism of structures which is onto then f is an isomorphism.

Question 3.
Let $\mathsf{L} = \{P, R\}$ where R is a binary relation symbol and P is a predicate. Describe all possible L-structures of size 2 upto isomorphism, i.e. give a list of L-structures of size 2 such that any L-structure is isomorphic to exactly one of them. Use the following steps:

(1) Write down all structures to L with universe $\{1, 2\}$.
(2) Divide them into $\cong$ equivalence classes.
(3) Show that every structure is isomorphic to one of these structures.

Question 4.
Suppose M is a structure, $A \subseteq M$. We let $\mathrm{Aut}(M/A)$ be the set of all automorphisms of M that fix A, i.e.

$$\{\sigma \in \mathrm{Aut}(M) \,|\forall x \in A \,(\sigma(x) = x)\}.$$

We let $\mathrm{Aut}(M/[A])$ be the set of all automorphisms of M that fix A setwise, i.e.

$$\left\{\sigma \in \mathrm{Aut}(M) \,\middle|\forall x \in A \left(\sigma(x) \in A \,\&\, \sigma^{-1}(x) \in A\right) \right\}$$

(1) Show that $\mathrm{Aut}(M/A)$ is a group with composition ($\circ$).
(2) Show that $\mathrm{Aut}(M/[A])$ is a group with composition.
(3) Show that $\mathrm{Aut}(M/A)$ is a normal subgroup of $\mathrm{Aut}(M/[A])$.