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MODEL THEORY – EXERCISE 10

To be submitted on Wednesday 22.06.2011 by 14:00 in the mailbox.

Definition. We introduce the following notation. Let L be a signature. Let Δ be a set of formulas.

(1) $M \Rrightarrow_\Delta N$ means: if $\varphi \in \Delta$ happens to be a sentence and $M \models \varphi$ then $N \models \varphi$. $M \equiv_\Delta N$ means that $M \Rrightarrow_\Delta N$ and $N \Rrightarrow_\Delta M$. If Δ contains all sentences then this just means $M \equiv N$.
(2) $f : M \to_\Delta N$ means that f is a map from M to N and in addition if $\varphi(\bar{x}) \in \Delta$ and $M \models \varphi(\bar{a})$ for some tuple $\bar{a}$ from M then $N \models \varphi(f[\bar{a}])$.
(3) If M is a structure and $A \subseteq M$ is a subset, we let $L(A)$ be L with new constant symbols c_a for elements $a \in A$. We denote (M, A) the $L(A)$ structure we get interpreting the constants in the obvious way. $\Delta(A)$ comes from the formulas in Δ by replacing any tuple of free variables $\bar{x}$ by all possible tuples $\bar{c}_a$ of the same length from A.
(4) Given a structure M, the diagram of M, $D(M)$ is the $L(M)$ theory

$$\{\varphi(\bar{a}) \,|\, \varphi \text{ atomic or negaion, } \bar{a} \in M, M \models \varphi(\bar{a})\}.$$

(5) We say that a formula φ is a $\exists\forall$ formula if it is of the form $\exists x_1 \ldots \exists x_n \psi(x_1, \ldots, x_n, \bar{y})$ where ψ is universal.
(6) We say that a structure M is generated by a subset $F \subseteq M$ if M is the minimal substructure of M that contains F. Equivalently, M is $\bigcup_{i \in \mathbb{N}} M_i$, where $M_0 = F$, and M_{i+1} is generated by M_i by applying all function symbols.

Question 1.
Prove the following:

(1) $f : M \to_\Delta N$ iff $(M, M) \Rrightarrow_{\Delta(M)} (N, f[M])$ (here $(N, f[M])$ is an $L(M)$ structure where we interpret a constant c_m as $f(m) \in N$).
(2) Let Δ be a set of sentences. Then $f : M \to_\Delta N$ iff $M \Rrightarrow_\Delta N$ and f is a map from M to N.
(3) $f : M \to N$ is a homomorphism iff $f : M \to_{at} N$ where at is the set of all atomic formulas.
(4) If Δ contains at and also the negation of all atomic formulas then f is an embedding (see Ex. 1).
(5) A homomorphism $f : M \to N$ is injective iff $f : M \to_\Delta N$ for the set $\Delta = \{x \neq y\}$.
(6) If Δ is a set of sentences closed under negation, then $M \Rrightarrow_\Delta N$ implies $M \equiv_\Delta N$.
(7) If Δ is a set of formulas closed under negation, then $f : M \to_\Delta N$ implies that we have iff in definition (2) above.

Question 2.

(1) In class you proved that a theory T is universal iff if T is preserved under substructures. Show that the following statements are equivalent:
   (a) T is existential, i.e. T can be axiomatized by existential sentences.
   (b) T is preserved under extensions, i.e. if $M \models T$ and $M \subseteq N$ then $N \models T$.

   Hint for (b) implies (a): Show that for all $\varphi \in T$, there is a finite set of existential sentences φ_i such that $T \models \bigvee \varphi_i$ and $\models \bigvee \varphi_i \to \varphi$. Use compactness.
(2) Conclude that the theory of Groups is neither existential nor universal.

Question 3.

(1) Suppose φ is a $\exists\forall$ sentence in the signature $L = \{<\}$, where $<$ is a binary relation symbol. Show that if φ is true in $(\mathbb{R}, <)$ then it is also true in $(\mathbb{N}, <)$.
(2) Is the converse true as well?

Question 4*.
Let L be some signature.
In Exercise 2, Question 3, (3), you proved that if φ is equational then φ is preserved under homomorphic images, products, and substructures (and you used this question in Exercise 9, Question 3). Now prove that if φ is preserved under homomorphic images, products, and substructures then φ is equivalent to a conjunction of equational sentences.

(1) Let Σ be the set of equational sentences ψ such that $\varphi \models \psi$. Show that is enough to show that $\Sigma \models \varphi$.
(2) Let A be any model of Σ. Show that it is a homomorphic image of a substructure of a product of models of φ. Use the following steps:
   (a) Show that there is a set C of structures, such that if $M \models \varphi$ and there is a finite nonempty set $F \subseteq M$ such that M is generated by F, then there is some $M' \in C$ such that $M' \cong M$.
   (b) Let D be the set of pairs (M, f) where $M \in C$ and $f : A \to M$ some function. Let $\mathfrak{M} = \prod \{M \,|(M, f) \in D\}$ (so there is some repetition). Define $F : A \to \mathfrak{M}$ by $F(a)(M, f) = f(a)$. Let $\mathfrak{N}$ be the structure generated by the image of F. Deduce that $\mathfrak{N} \models \varphi$.
   (c) Define a homomorphism $G : \mathfrak{N} \to A$ that satisfies $G(F(a)) = a$ and conclude.