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# MODEL THEORY – EXERCISE 11

To be submitted on Wednesday 29.06.2011 by 14:00 in the mailbox.

**Definition.** We introduce the following notation. Let $L$ be a signature. Let $\Delta$ be a set of formulas.

(1) The *Prüfer p-group* $\mathbb{Z}_{p^\infty}$, for a prime number $p$ is the multiplicative group of all $p^n$th roots of unity in $\mathbb{C}^\times$ for all $n < \omega$.
(2) The *fundamental theorem for finitely generated abelian groups* states that every finitely generated abelian group $G$ is isomorphic to a direct sum of primary cyclic groups and infinite cyclic groups. A primary cyclic group is one whose order is a power of a prime.
(3) An abelian group $G$ is divisible if for every $n < \omega$ and $x \in G$ there is some $y \in G$ such that $ny = x$.
(4) An abelian group $G$, equipped with a linear order $<$ is an *ordered abelian group* iff it satisfies $\forall xyu\,(x < y \Rightarrow x + u < y + u)$.
(5) An abelian group $G$ is said to be *orderable* iff there exists some $<$ such that $(G, <)$ is an ordered abelian group.
(6) A group is called *locally finite* if every finitely generated subgroup is finite.
(7) A class of structures $K$ is called *hereditary* if whenever $M \in K$ and $N \subseteq M$ (a substructure) $N \in K$.

**Question 1.**
Let $p$ be a prime number and $n$ a positive natural number.

(1) Prove that the group $G$ of all $p^n$ roots of unity in $\mathbb{C}^\times$ is cyclic of order $p^n$.
(2) Conclude that the Prüfer $p$-group is a union $\bigcup_{i<\omega} G_i$ of finite cyclic groups of order $p^i$ such that $G_i \leq G_{i+1}$.
(3) Prove that the Prüfer $p$-group is a divisible abelian group.
(4) Conclude that if $G$ is an abelian group, then $G$ can be embedded in a divisible abelian group.

**Question 2.**
Suppose $G$ is an abelian group. Show that $G$ is orderable iff $G$ is torsion free.

**Question 3.**

(1) Suppose $I$ and $J$ are 2 linear orders and that $J$ is infinite. Show that there is embedding of $I$ into a model of $Th\,(J)$.
(2) In particular show conclude that every linear order can be embedded into a dense linear order.

**Question 4.**
Show that the class of locally finite groups is hereditary, but not elementary.