Prof. Dr. Salma Kuhlmann Dr. Itay Kaplan

MODEL THEORY – EXERCISE 2

To be submitted on Wednesday 27.04.2011 by 14:00 in the mailbox.

Definition.
Suppose L is a signature, $L' \subseteq L$, M, N are L-structures.

(1) We say that M is a *substructure* of N, denoted by $M \subseteq N$ if the universe of M is a subset of the universe of N and $\mathrm{id} : M \to N$ (the identity map) is an embedding.
(2) We say that M is the L'-*reduct* of N, denoted by $M = N \upharpoonright L'$ if M is the L'-structure whose universe is the same universe as N and all symbols in L' are interpreted as in N.
(3) Given a formula φ, we write $\varphi(x_0, x_1, \ldots, x_{n-1})$ to indicate that the set of free variables of φ is contained in $\{x_0, \ldots, x_{n-1}\}$.
(4) We call a formula φ a *sentence* if it has no free variables.
(5) We call a formula φ is *quantifier free* if it does not contain the quantifiers $\exists, \forall$.
(6) We call a formula φ universal if it of the form $\forall x_0 \forall x_1 \ldots \forall x_{n-1} \psi$ where ψ is quantifier free.
(7) We call a formula φ existential if it is of the form $\exists x_0 \exists x_1 \ldots \exists x_{n-1} \psi$ where ψ is quantifier free.
(8) We call a formula φ equational if it is of the form $\forall x_0 \ldots \forall x_{n-1} \psi$ where ψ is atomic.
(9) For 2 L-sentences φ and ψ, we write $\psi \models \varphi$ if for every L-structure M, $M \models \psi \Rightarrow M \models \varphi$.
(10) We say that two L-sentences φ and ψ are *elementarily equivalent* if for every L-structure M, $M \models \varphi$ iff $M \models \psi$.

Question 1.
In the following clauses you are given a signature and a mathematical statement in English. Write this statement as a sentence in the signature.

(1) $L = \{f, +, -, |\,|, 0, <\}$ where $f, |\,|$ are unary function symbols, $+, -$ are binary function symbols, 0 is a constant and $<$ is a binary relation symbol. The function f is a continuous function when we think of it as a function from $\mathbb{R}$ to $\mathbb{R}$, and interpret all symbols as in $\mathbb{R}$.
(2) $L = \{f, <, 0\}$ where $f, <, 0$ are as in (1). The function f is a continuous function in 0 (when we think of it as a function from $\mathbb{R}$ to $\mathbb{R}$).
Solution: just recall the topological definition of continuous: for every open interval (a, b) containing $f(0)$, $f^{-1}(a, b)$ contains an interval around 0.
(3) $L = \{<, c_0, c_1\}$ where $<$ is a binary relation symbol, c_0, c_1 are constants. $<$ is a dense linear order (between every 2 elements there is a third one), and c_0 is the first element, c_1 is the last element.
(4) $L = \{S\}$ where S is a unary relation symbol. The universe contains exactly 5 elements, exactly 3 of them are in S.
(5) $L = \{R\}$ where R is a binary relation symbol. R is a graph of a function.
(6) $L = \{R\}$ as in (5). R is a graph of a surjective function.

(7) $L = \{+, \cdot, 0, 1\}$ where $+, \cdot$ are binary function symbols, $0, 1$ are constants (this is the signature of rings). For every polynomial of degree 5 there exists a root.

Question 2.

Suppose $M \subseteq N$ are 2 L-structures (so M is a substructure of N).

(1) Suppose $\varphi(x_0, \dots, x_{n-1})$ is a quantifier free formula, and $a_0, \dots, a_{n-1} \in M$. Show that $M \models \varphi[a_0, \dots, a_{n-1}]$ iff $N \models \varphi[a_0, \dots, a_{n-1}]$.
Solution: prove first by induction on terms that for all terms $t(x_0, \dots, x_{m-1})$ and all $b_0, \dots, b_{m-1} \in M$, $t^M[b_0, \dots, b_{m-1}] = t^N[b_0, \dots, b_{n-1}]$. Then show it by induction to all quantifier free formulas.

(2) Suppose $\varphi(x_0, \dots, x_{n-1})$ is a universal formula (so of the form $\forall y_0 \dots \forall y_{m-1} \psi(y_0, \dots, y_{m-1}, x_0, \dots, x_{n-1})$).
Show that if $N \models \varphi[a_0, \dots, a_{n-1}]$ then $M \models \varphi[a_0, \dots, a_{n-1}]$.

(3) Suppose $\varphi(x_0, \dots, x_{n-1})$ is an existential formula. Show that if $M \models \varphi[a_0, \dots, a_{n-1}]$ then $N \models \varphi[a_0, \dots, a_{n-1}]$.

(4) Let $L = \{P\}$ where P is a unary relation symbol. Show that the sentence $\forall x P(x)$ is not elementarily equivalent to an existential sentence.
Solution: suppose it was, to ψ. Let A be the structure with universe $\{0\}$ and $P = \{0\}$, and let B be the structure with universe $\{0, 1\}$ and $P = \{0\}$. Then $A \subseteq B$ and $A \models \forall x P(x)$. If ψ was equivalent to it, then $B \models \forall x P(x)$ too (by 3) – a contradiction.

Question 3.

(1) Suppose $f : A \to B$ is a homomorphism between two L-structures. Show that the image of f is a substructure of B.

(2) Let $\{A_i \mid i \in I\}$ be structures for a signature L, and let $L' \subseteq L$. Show that $\left(\prod_i A_i\right) \upharpoonright L' = \prod_i (A_i \upharpoonright L')$.

(3) Show that if φ is equational then φ is preserved under homomorphic images, products, and substructures.
Explanation: you need to show that
   (a) If $A_i \models \varphi$ for all $i \in I$, then $\prod A_i \models \varphi$.
   Solution: By induction on a term $t(x_0, \dots, x_{n-1})$, show that for all $\bar{a} = \langle \bar{a}_i \mid i \in I \rangle \in \prod A_i$, $t^{\prod A_i}(\bar{a}) = \langle t^{A_i}(\bar{a}_i) \mid i \in I \rangle$.
   (b) If $A \subseteq B$ and $B \models \varphi$ then $A \models \varphi$.
   (c) If $f : A \to B$ is a surjective homomorphism and $A \models \varphi$ then $B \models \varphi$.