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MODEL THEORY – EXERCISE 3

To be submitted on Wednesday 04.05.2011 by 14:00 in the mailbox.

Definition.
Suppose L is a signature.

(1) For a set of sentences Σ and for a structure M, we write $M \models \Sigma$ for $M \models \varphi$ for every $\varphi \in \Sigma$.
(2) For a set of sentences Σ in L, and an L-sentence φ, we write $\Sigma \models \varphi$ when for all L-structures $M \models \Sigma \Rightarrow M \models \varphi$.
(3) For two sets of formulas in free variables $\bar{x} = (x_0, \dots, x_{n-1})$, $\Psi(\bar{x})$ and $\Theta(\bar{x})$, and a set of sentences Σ, we say that $\Psi(\bar{x})$ and $\Theta(\bar{x})$ are *logically equivalent modulo* Σ if for every $M \models \Sigma$, and every $\bar{a} = (a_0, \dots, a_{n-1})$, $M \models \psi[\bar{a}]$ for all $\psi(\bar{x}) \in \Psi(\bar{x})$ iff $M \models \theta[\bar{a}]$ for all $\theta(\bar{x}) \in \Theta(\bar{x})$.
(4) If Σ in (4) is empty, we say that Ψ and Θ are logically equivalent.
(5) Given a structure M and a subset $A \subseteq M$, a subset $X \subseteq M^n$ is said to be *definable over* A if there is an L-formula $\varphi(x_0, \dots, x_{n-1}, y_0, \dots, y_{k-1})$ and parameters from A – $b_0, \dots, b_{k-1} \in A$ such that

$$X = \{(a_0, \dots, a_{n-1}) \in M^n \,|M \models \varphi[a_0, \dots, a_{n-1}, b_0, \dots, b_{k-1}]\}.$$

Question 1.
Let L be a signature that contains at least one constant symbol c. Let M be an L-structure.

(1) Let $t(x_0, \dots, x_{n-1})$ be a term in L (this notation means that t uses variables only from $x_0, \dots, x_{n-1}$). Denote by $t(c)$ the term induced by replacing every appearance of x_0 with c. Show that $t(c)^M[a_1, \dots, a_{n-1}] = t\left[c^M, a_1, \dots, a_{n-1}\right]$ for every $a_1, \dots, a_{n-1} \in M$.
(2) Now suppose that $\varphi(x_0, \dots, x_{n-1})$. Denote by $\varphi(c)$ the formula induced by replacing every *free* appearance of x_0 by c. Show that $M \models \varphi(c)[a_1, \dots, a_{n-1}]$ iff $M \models \varphi\left[c^M, a_1, \dots, a_{n-1}\right]$ for every $a_1, \dots, a_{n-1} \in M$.
(3) Now let $L' \subseteq L$, $M' = M \restriction L'$. Suppose $\varphi(x_0, \dots, x_{n-1})$ is an L' formula, then for all $a_0, \dots, a_{n-1} \in M$, $M' \models \varphi[a_0, \dots, a_{n-1}]$ iff $M \models \varphi[a_0, \dots, a_{n-1}]$.
(4) Prove the following claim:
Assume that c does not appear in the sentences φ and ψ, and that φ is of the form $\exists x \alpha(x)$. Show that $\varphi \models \psi$ iff $\alpha(c) \models \psi$.
(5) Show that the claim in 4 is not true if we allow c to appear in φ.
Do the same for ψ.

Question 2.
A set of sentences Σ is called independent iff for no $\varphi \in \Sigma$, $\Sigma \setminus \{\varphi\} \models \varphi$.

(1) Show that if Σ is finite, then it has an independent equivalent subset (i.e. logically equivalent to Σ).
(2) Find an example of an infinite Σ without an independent equivalent subset.

(3) Now assume that $\Sigma = \{\alpha_i \,|\, i \in \mathbb{N}\}$. Show that there is some independent equivalent Σ' (not necessarily being a subset).

Question 3.
Suppose $\varphi(\bar{x})$ is a quantifier free formula. Show that it can be written in disjunctive normal form, i.e. that $\varphi(\bar{x})$ is logically equivalent to a formula $\psi(\bar{x})$ where $\psi(\bar{x}) = \bigvee_{i<n} \bigwedge_{j<k} \alpha_{i,j}(\bar{x})$ where $\alpha_{i,j}$ is atomic or negation of atomic.
Hint: let Γ be the set of all atomic formulas appearing in φ (so it is finite). For every structure M, and tuple $\bar{a}$ (in the length of $\bar{x}$), let $f_{M,\bar{a}} : \Gamma \to \{T, F\}$ satisfy $f_{M,\bar{a}}(\alpha) = T$ iff $M \models \alpha[\bar{a}]$. Show by induction on φ that if $f_{M,\bar{a}_1} = f_{M,\bar{a}_2}$ then $M_1 \models \varphi[\bar{a}_1]$ iff $M_2 \models \varphi[\bar{a_2}]$. For each function $f : \Gamma \to \{T, F\}$, let φ^f be this truth value (if f does not appear as $f_{M,\bar{a}}$, then choose φ^f arbitrarily).
Let $A = \left\{f : \Gamma \to \{T, F\} \,\middle|\, \varphi^f = T\right\}$, show that φ is equivalent to $\bigvee_{f \in A} \bigwedge \alpha^{f(\alpha)}(\bar{x})$ where $\alpha^T = \alpha$ and $\alpha^F = \neg\alpha$.

Question 4.

(1) Show that if M is a structure, $X \subseteq M$ is definable over $A \subseteq M$, and σ is an automorphism of M fixing A (i.e. $\sigma(a) = a$ for all a) then $\sigma(X) = X$.
Let $L = \{<\}$. Recall that a linear order is called *dense* if for any $a < b$ there exists c such that $a < c < b$.
(2) Write down a list of axiom in L for the theory DLO – dense linear order without first and last element.
(3) Show that $(\mathbb{Q}, <)$ is a model of this theory.
(4) Describe all definable subsets of $\mathbb{Q}$ over $\emptyset$ that are definable without quantifiers
Hint: use (1).
(5) Describe all definable subsets of $\mathbb{Q}$ over $\mathbb{Q}$ that are definable without quantifiers (i.e. that the formulas defining them are quantifier free).
Hint: try to guess what the answer is, and then prove it by induction on the formula.