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MODEL THEORY – EXERCISE 3

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Definition.

Suppose L is a signature.

(1) For a set of sentences Σ in L, and an L-sentence φ, we write $\Sigma \models \varphi$ when for all L-structures $M \models \Sigma \Rightarrow M \models \varphi$.
(2) For a set of sentences Σ and for a structure M, we write $M \models \Sigma$ for $M \models \varphi$ for every $\varphi \in \Sigma$.
(3) For a set of sentences Σ, let $\Sigma^{\models}$ be the *deductive closure* of Σ, i.e. $\{\varphi \,|\Sigma \models \varphi\}$.
(4) For two sets of formulas in free variables $\bar{x} = (x_0, \dots, x_{n-1})$, $\Psi(\bar{x})$ and $\Theta(\bar{x})$, and a set of sentences Σ, we say that $\Psi(\bar{x})$ and $\Theta(\bar{x})$ are *logically equivalent modulo* Σ if for every $M \models \Sigma$, and every $\bar{a} = (a_0, \dots, a_{n-1})$, $M \models \psi[\bar{a}]$ for all $\psi(\bar{x}) \in \Psi(\bar{x})$ iff $M \models \theta[\bar{a}]$ for all $\theta(\bar{x}) \in \Theta(\bar{x})$.
(5) If Σ in (4) is empty, we say that Ψ and Θ are logically equivalent.
(6) Given a structure M and a subset $A \subseteq M$, a subset $X \subseteq M^n$ is said to be *definable over* A if there is an L-formula $\varphi(x_0, \dots, x_{n-1}, y_0, \dots, y_{k-1})$ and parameters from A – $b_0, \dots, b_{k-1} \in A$ such that

$$X = \{(a_0, \dots, a_{n-1}) \in M^n \,|M \models \varphi[a_0, \dots, a_{n-1}, b_0, \dots, b_{k-1}]\}.$$

Question 1.

Let L be a signature that contains at least one constant symbol c. Let M be an L-structure.

(1) Let $t(x_0, \dots, x_{n-1})$ be a term in L (this notation means that t uses variables only from $x_0, \dots, x_{n-1}$). Denote by $t(c)$ the term induced by replacing every appearance of x_0 with c. Show that $t(c)^M[a_1, \dots, a_{n-1}] = t\left[c^M, a_1, \dots, a_{n-1}\right]$ for every $a_1, \dots, a_{n-1} \in M$.
Solution: by induction on t.
(2) Now suppose that $\varphi(x_0, \dots, x_{n-1})$. Denote by $\varphi(c)$ the formula induced by replacing every *free* appearance of x_0 by c. Show that $M \models \varphi(c)[a_1, \dots, a_{n-1}]$ iff $M \models \varphi\left[c^M, a_1, \dots, a_{n-1}\right]$ for every $a_1, \dots, a_{n-1} \in M$.
Solution: by induction on the formula φ.
(3) Now let $L' \subseteq L$, $M' = M \restriction L'$. Suppose $\varphi(x_0, \dots, x_{n-1})$ is an L' formula, then for all $a_0, \dots, a_{n-1} \in M$, $M' \models \varphi[a_0, \dots, a_{n-1}]$ iff $M \models \varphi[a_0, \dots, a_{n-1}]$.
Solution: by induction on φ (first you need some induction on terms).
(4) Prove the following claim:
Assume that c does not appear in the sentences φ and ψ, and that φ is of the form $\exists x \alpha(x)$. Show that $\varphi \models \psi$ iff $\alpha(c) \models \psi$.
Solution: Left to right: if $M \models \alpha(c)$ then $M \models \alpha\left[c^M\right]$ by 3, so $M \models \exists x \alpha(x)$ so also $M \models \psi$. For the other direction, suppose $\alpha(c) \models \psi$. Let $M \models \exists x \alpha(x)$. So there is some $b \in M$ such that $M \models \alpha[b]$. Let $M' = M \restriction L \backslash \{c\}$. Note that $M' \models \alpha[b]$ by 2. Let M'' be the L-structure

induced by M' by declaring $c^{M'} = b$. Then $M'' \models \alpha[b]$ as well, but also $M'' \models \alpha(c)$ by 2. So $M'' \models \psi$, but then $M' \models \psi$ by 2, so $M \models \psi$ by 2.

(5) Show that the claim in 4 is not true if we allow c to be in φ. Do the same for ψ.
Solution: 1. let $\alpha = x \neq c$, $\psi = \exists x (x \neq x)$. 2. let $L = \{c, d\}$ (d a constant), and $\alpha = x \approx d$ and $\psi = c \approx d$.

Question 2.
Suppose $\Sigma, \Sigma_1, \Sigma_2$ are sets of sentences in L.

(1) Show that the deductive closure of Σ is deductively closed, i.e. $\left(\Sigma^{\models}\right)^{\models} = \Sigma^{\models}$.
(2) Show that the following are equivalent:
  (a) Σ_1 and Σ_2 are logically equivalent modulo Σ.
  (b) $(\Sigma \cup \Sigma_1)^{\models} = (\Sigma \cup \Sigma_2)^{\models}$.
  (c) For any structure M such that $M \models \Sigma$, $M \models \Sigma_1$ iff $M \models \Sigma_2$.
  Solution: a iff c is immediate from the definition. c implies b: assume $\Sigma \cup \Sigma_1 \models \varphi$, take any model of $\Sigma \cup \Sigma_2$ then it is also a model of $\Sigma \cup \Sigma_1$ so of φ. b implies c is clear.
(3) Suppose $\varphi_0(x)$, $\varphi_1(x)$, ... , $\varphi_n(x)$, $\psi_0(x)$, $\psi_1(x)$, ... , $\psi_k(x)$ are finitely many formulas. Find a sentence α such that $\{\varphi_i(x) \mid i < n\}$ and $\{\psi_i(x) \mid i < k\}$ are logically equivalent modulo Σ iff $\Sigma \models \alpha$.
Solution: $\alpha = \forall x \left(\bigwedge_{i<n} \varphi_i(x) \leftrightarrow \bigwedge_{i<k} \psi_i(x)\right)$.

Question 3.
A set of sentences Σ is called independent iff for no $\varphi \in \Sigma$, $\Sigma \setminus \{\varphi\} \models \varphi$.

(1) Show that if Σ is finite, then it has an independent equivalent subset.
Solution: just take minimal equivalent subset.
(2) Find an example of an infinite Σ without an independent equivalent subset.
Solution: let $\Sigma = \{P_0, P_0 \wedge P_1, P_0 \wedge P_1 \wedge P_2, \ldots\}$ in the language $\{P_i\}$ where P_i are 0-relation symbols (you can replace by $R_i(c)$ for a predicate R and a constant c).
(3) Now assume that $\Sigma = \{\alpha_i \mid i \in \mathbb{N}\}$. Show that there is an independent equivalent Σ' (not necessarily being a subset).
Solution: suppose α_{i_0} is the first sentence that isn't always true (if there is none, then let $\Sigma' = \{\alpha_0\}$). Let i_1 be the first i such that $\alpha_{i_0} \not\models \alpha_{i_1}$. Let i_2 be the first i such that $\alpha_{i_0} \wedge \alpha_{i_1} \not\models \alpha_{i_2}$. Continue in this way (let i_{n+1} be the first i such that $\alpha_{i_0} \wedge \cdots \wedge \alpha_{i_n} \not\models \alpha_{n+1}$). Let $\Sigma' = \{\alpha_{i_0} \wedge \cdots \wedge \alpha_{i_n} \to \alpha_{i_{n+1}}\} \cup \{\alpha_{i_0}\}$. So Σ' is equivalent with Σ (that's obvious). Moreover, Σ' is independent, because if we remove α_{i_0}, let M be a structure where α_{i_0} is false, and it will satisfy $\alpha_{i_0} \wedge \cdots \wedge \alpha_{i_n} \to \alpha_{i_{n+1}}$ for all n. If we remove $\alpha_{i_0} \wedge \cdots \wedge \alpha_{i_n} \to \alpha_{i_{n+1}}$ for some n, let M be a structure where $\alpha_{i_0} \wedge \cdots \wedge \alpha_{i_n}$ holds but $\alpha_{i_{n+1}}$ does not, and it will be a model of all sentences except this one.

Question 4.
Suppose $\varphi(\bar{x})$ is a quantifier free formula. Show that it can be written in disjunctive normal form, i.e. that $\varphi(\bar{x})$ is logically equivalent to a formula $\psi(\bar{x})$ where $\psi(\bar{x}) = \bigvee_{i<n} \bigwedge_{j<k} \alpha_{i,j}(\bar{x})$ where $\alpha_{i,j}$ is atomic or negation of atomic.
Hint: let Γ be the set of all atomic formulas appearing in ψ (so it is finite). For

every structure M, and tuple $\bar{a}$ (in the length of $\bar{x}$), let $f_{M,\bar{a}} : \Gamma \to \{T, F\}$ be $f_{M,\bar{a}}(\alpha) = T$ iff $M \models \alpha[\bar{a}]$. Show that if $f_{M,\bar{a}_1} = f_{M,\bar{a}_2}$ then $M_1 \models \alpha[\bar{a}_1]$ iff $M_2 \models \alpha[\bar{a_2}]$. Prove this by induction on ψ. For each function $f : \Gamma \to \{T, F\}$, let ψ^f be this truth value (if f does not appear as $f_{M,\bar{a}}$, then choose ψ^f arbitrarily). Let $A = \left\{f : \Gamma \to \{T, F\} \,\middle|\, \psi^f = T\right\}$, show that ψ is equivalent to $\bigvee_{f \in A} \bigwedge \alpha^{f(\alpha)}(\bar{x})$ where $\alpha^T = \alpha$ and $\alpha^F = \neg\alpha$.

Question 5.

(1) Show that if M is a structure, $X \subseteq M$ is definable over $A \subseteq M$, and σ is an automorphism of M fixing A (i.e. $\sigma(a) = a$ for all a) then $\sigma(X) = X$.
(2) Let $L = \{<\}$. Recall that a linear order is called *dense* if for any $a < b$ there exists c such that $a < c < b$.
(3) Write down a list of axiom in L for the theory DLO – dense linear order without first and last element.
(4) Show that $(\mathbb{Q}, <)$ is a model of this theory.
(5) Describe all definable subsets of $\mathbb{Q}$ over $\emptyset$ that are definable without quantifiers
Hint: use (1).
Solution: $\mathbb{Q}$, $\emptyset$. Why? if $X \subseteq \mathbb{Q}$ is definable, and $a \in X$, $b \in \mathbb{Q}$, then there exists some automorphism taking a to b.
(6) Describe all definable subsets of $\mathbb{Q}$ over $\mathbb{Q}$ that are definable without quantifiers (i.e. that the formulas defining them are quantifier free).
Hint: try to guess what the answer is, and then prove it by induction on the formula.
Solution: finite union of points and intervals. Why? obvious for atomic formulas, and in general just by induction.
(7) Bonus: is this a complete theory?