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MODEL THEORY – EXERCISE 6

To be submitted on Wednesday 25.05.2011 by 14:00 in the mailbox.

Definition.

(1) For a set of ordinals, $s \subseteq \mathbf{On}$, let the order type of s, $\mathrm{otp}(s)$ be the unique ordinal with which s is isomorphic.
(2) Suppose α is a ordinal. A subset $B \subseteq \alpha$ is called *cofinal in* α (or unbounded) if $\alpha = \bigcup \{\gamma + 1 \,|\gamma \in B\}$ (this means that for every $\beta < \alpha$, there is some $\gamma \in B$ such that $\beta \leq \gamma$).
(3) For an ordinal α, the *cofinality* of α, $\mathrm{cf}(\alpha)$, is $\min\{\mathrm{otp}(B)\,|B \subseteq \alpha \text{ is cofinal}\}$.
(4) A cardinal λ is called *regular* if $\mathrm{cf}(\lambda) = \lambda$.

Question 1.
Prove the Cantor-Bernstein theorem: if A, B are sets, and there is some injective function $f : A \to B$ and some injective function $g : B \to A$, then $|A| = |B|$.

Question 2.
Let α be an ordinal.

(1) Show that if α is a successor ordinal, then $\mathrm{cf}(\alpha) = 1$.
(2) Show that $\mathrm{cf}(\alpha)$ is always a cardinal.
(3) Conclude that $\mathrm{cf}(\alpha)$ can be defined by $\min\{|B|\,|B \subseteq \alpha \text{ is cofinal}\}$.
Now let λ be an infinite cardinal.
(4) Show that $\lambda \geq cf(\lambda)$.
(5) Show that $cf(cf(\lambda)) = cf(\lambda)$.
(6) Show that $\aleph_0$ is regular, and that κ^+ is regular for all infinite κ.
(7) Show that for limit ordinal α, $\mathrm{cf}(\aleph_\alpha) = \mathrm{cf}(\alpha)$ and give an example of an irregular cardinal.

Question 3.

(1) Let λ be a cardinal. Show that if $\langle \kappa_i \,|i < \lambda\rangle$ is a sequence of λ cardinals, such that $i < j < \lambda \Rightarrow \kappa_i < \kappa_j$, then $\sum_{i<\lambda} \kappa_i < \prod_{i<\lambda} \kappa_i$ (where $\sum_{i<\kappa} \kappa_i$ is the cardinality of the disjoint union $\coprod \kappa_i$, and $\prod \kappa_i$ is the cardinality of the Cartesian product).
Hint: try to find a diagonalizing argument, as in the proof of $\kappa < 2^\kappa$. Note that for $i < \lambda$, $\sum_{j\leq i} \kappa_i = \kappa_i$.
(2) Conclude that $\mathrm{cf}(2^\kappa) > \kappa$ (note that this generalizes the fact proved in class that $2^\kappa > \kappa$).
Hint: deal with 2 cases: 2^κ is a successor or limit cardinal.

Question 4.
Let K be a field.

(1) Show that the cardinality of the algebraic elements over K in some field extension F is bounded by $|K| + \aleph_0$.
(2) Let V be an infinite vector space over K, and let B be a basis for V. Show that $|B| + |K| + \aleph_0 = |V|$.

(3) Show that the cardinality of the set of irrational real numbers is $2^{\aleph_0}$.
(4) Show that the cardinality of the set of real transcendental numbers is $2^{\aleph_0}$ (i.e. elements that are in $\mathbb{R}$ but not algebraic over $\mathbb{Q}$).