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MODEL THEORY – EXERCISE 8

To be submitted on Wednesday 08.06.2011 by 14:00 in the mailbox.

Definition.

(1) For a set X and a number $n < \omega$, let $X^{[n]}$ be the set of subsets of X of size n.
(2) A graph is a structure $G = (V, E)$ where V is a set of vertices and the edge relation E is a binary relation which is symmetric ($xEy \to yEx$) and anti-reflexive ($\neg xEx$).
(3) The Finiteness Theorem / The Compactness Theorem: if Σ is a set of sentence such that each finite subset of it is consistent (has a model) then Σ has a model.
(4) Let $n < \omega$. We say that a graph G is n-colorable, if there exists a function $C : V \to n$ such that if $a, b \in V$ and aEb then $C(a) \neq C(b)$.
(5) A class of L-structures K is called elementary if there exists a set of sentences Σ such that $K = Mod(\Sigma)$.

Question 1.

(1) Let $L = \{<\}$. Show that the class of well orderings is not elementary.
(2) Show that the class of all finite sets (in the signature $L = \approx$) is not elementary.
(3) Let $L = \{+, \cdot, 0, 1, <\}$, and let $T = Th(\mathbb{N}, +, \cdot, 0, 1, <)$. Show that there exists a model M of T with an element c which is greater than all natural numbers (i.e. $c > 1^M, (1+1)^M$ etc.)
(4) Show that in the model constructed in (3), there is no minimal such c.
(5) Let $T = Th(\mathbb{R}, +, \cdot, 0, 1, <)$. Show that there is a model $M \models T$ with an element $0 < \varepsilon \in M$ which is infinitesimal: for every positive integer n, $\varepsilon < \left(1/(1 + \ldots + 1)^M\right)$ where the 1 is summed n-times.

Question 2.

The infinite Ramsey Theorem states as follows: Suppose V is an infinite set and $C : V^{[2]} \to \{0, 1\}$. Then there exists an infinite subset $U \subseteq V$ and $i \in \{0, 1\}$ such that $C(\{x, y\}) = i$ for all $x, y \in U$ (in other words, $C \upharpoonright U^{[2]}$ is constant).
You may think of C as a coloring function (of pairs from V), and then U is monochromatic.
The finite Ramsey Theorem states as follows: For all $k < \omega$ there exists some $n < \omega$ such that if $|V| = n$, and $C : V^{[2]} \to \{0, 1\}$ then there exists some $U \subseteq V$ of size k which is monochromatic.
Remark: this is actually the Ramsey Theorem for coloring of pairs in 2 colors.

(1) Prove the infinite Ramsey theorem.
(2) Deduce the finite Ramsey Theorem from the infinite one using the Compactness Theorem. (You should try to do this clause even if you could not solve (1)).

Question 3.
Show that if $G = (V, E)$ is an infinite graph such that every finite sub graph of it is n-colorable then G is n-colorable.

Question 4.
Show that the following are equivalent:

(1) The Compactness Theorem.
(2) Let T_1, T_2 be sets of L-sentences. Assume that for every L-structure M, M is a model of T_1 iff M is not a model of T_2. Then there are some finite $\Sigma_1 \subseteq T_1$, $\Sigma_2 \subseteq T_2$ such that $\Sigma_1 \equiv T_1$, $\Sigma_2 \equiv T_2$ (i.e. $\Sigma_1 \models T_1$, $\Sigma_2 \models T_2$).
(3) Let T_1, T_2 be sets of L-sentences. Assume that T_2 is finite and that $T_1 \equiv T_2$. Then T_1 is finitely axiomatizable (i.e. there is some finite $\Sigma \subseteq T_1$ such that $\Sigma \equiv T_1$).