Prof. Dr. Salma Kuhlmann Dr. Itay Kaplan

MODEL THEORY – EXERCISE 9

To be submitted on Wednesday 15.06.2011 by 14:00 in the mailbox.

Definition.

(1) A *Boolean Algebra* is a structure to the language $\{\wedge, \vee, \neg, 0, 1\}$ satisfying the following axioms:
   (a) associativity $a \wedge (b \wedge c) = (a \wedge b) \wedge c$, $a \vee (b \vee c) = (a \vee b) \vee c$.
   (b) commutativity $a \wedge b = b \wedge a$, $a \vee b = b \vee a$.
   (c) absorption law $a \vee (a \wedge b) = a$, $a \wedge (a \vee b) = a$.
   (d) distributivity $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$, $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$
   (e) $a \wedge \neg a = 0$, $a \vee \neg a = 1$.
(2) If B is a Boolean Algebra, then it gives rise to the following (partial) ordering: $x \leq y \Leftrightarrow x \wedge y = x$ (so $x < y \Leftrightarrow x \leq y \& x \neq y$).
(3) Let B be a Boolean Algebra. A set $F \subseteq B$ is called a *filter* on B if the following holds:
   (a) $0 \notin F$.
   (b) $F \neq \emptyset$.
   (c) If $a, b \in F$ then $a \wedge b \in F$.
   (d) If $a \in F$ and $a \leq b$ then $b \in F$.
(4) F is said to be a *principle filter* if there exists some $x \in B$ such that $F = \{y \in B \,|x \leq y\}$.
(5) F is said to be an *ultrafilter* if for every $x \in B$, $x \in F$ or $\neg x \in F$.

Question 1.
Let B a Boolean algebra.

(1) Show that $<$ is an ordering with maximum 1 and minimum 0. Show that $f : B \to B'$ is an isomorphism of Boolean Algebras iff f is an isomorphism between $(B, <)$ and $(B', <)$.
(2) Let $S(B)$ be the set of all ultrafilters on B. Show that every filter on B can be extended to an ultrafilter on B, and thus $S(B)$ is not empty.
    This is a Boolean algebra (of subsets of B), and it is called the *Stone space* of B.
(3) For $x \in B$, let $[x] = \{y \in S(B)\,|x \in y\}$. Check that $\{[x]\,|x \in B\}$ defines a closed-open basis for a topology on $S(B)$.
(4) Show that $S(B)$ is compact Hausdorff space.
(5) Check that every closed-open set in $S(B)$ is of the form $[x]$ for some $x \in B$.
    Let B' be the set of all closed-open sets in $S(B)$.
(6) Let $f : B \to B'$ be $f(x) = [x]$. Show that f is an isomorphism of Boolean Algebras.
    This is *Stone's representation theorem.*
(7) Deduce that B can is isomorphic to a sub-Boolean algebra of the power set $\mathcal{P}(X)$ for some set X.

Question 2.

Let L be a signature. Let B be *Lindenbaum–Tarski* algebra of L be the set equivalence classes (under elementary equivalence) of sentences of L.

(1) We give B a structure of a Boolean Algebra in the obvious way (for instance, given $[\varphi], [\psi]$, we define $[\varphi] \wedge [\psi] = [\varphi \wedge \psi]$). Make sure it is well defined.
(2) For a complete theory T, let $F(T) := \{[\varphi] \,|\varphi \in T\}$. Prove that $F(T)$ is in $S(B)$.
(3) Prove that the following are equivalent:
    (a) F is an isomorphism.
    (b) The finiteness / compactness theorem.

Question 3.

Recall exercise 2. All the axioms of Boolean algebra are equational, and hence by that Question 3, (3) there, substructures, homomorphic images and products of Boolean algebras are also Boolean algebras. Let 2 be the trivial Boolean algebra: the universe is just $\{0, 1\}$ and everything is defined by the axioms.

(1) Let X be a set. Show that the algebra on $\mathcal{P}(X)$ is isomorphic to 2^X (i.e. the product $\prod \langle 2_x \,| x \in X \rangle$ where $2_x = 2$ for all $x \in X$).
(2) Show that the theory of Boolean algebras (i.e. the set of axioms in the definitions) is elementarily equivalent to $\Sigma := \{\varphi \,| 2 \models \varphi, \varphi \text{ is equational}\}$. (See Ex. 2).
    Hint: use Exercise 2, Question 3 (3).