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MODEL THEORY – EXERCISE 9

To be submitted on Wednesday 15.06.2011 by 14:00 in the mailbox.

Definition.

(1) A *Boolean Algebra* is a structure to the language $\{\wedge, \vee, \neg, 0, 1\}$ satisfying the following axioms:
  (a) associativity $a \wedge (b \wedge c) = (a \wedge b) \wedge c$, $a \vee (b \vee c) = (a \vee b) \vee c$.
  (b) commutativity $a \wedge b = b \wedge a$, $a \vee b = b \vee a$.
  (c) absorption law $a \vee (a \wedge b) = a$, $a \wedge (a \vee b) = a$.
  (d) distributivity $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$, $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$
  (e) $a \wedge \neg a = 0$, $a \vee \neg a = 1$.

(2) If B is a Boolean Algebra, then it gives rise to the following (partial) ordering: $x \leq y \Leftrightarrow x \wedge y = x$ (so $x < y \Leftrightarrow x \leq y \& x \neq y$).

(3) Let B be a Boolean Algebra. A set $F \subseteq B$ is called a *filter* on B if the following holds:
  (a) $0 \notin F$.
  (b) $F \neq \emptyset$.
  (c) If $a, b \in F$ then $a \wedge b \in F$.
  (d) If $a \in F$ and $a \leq b$ then $b \in F$.

(4) F is said to be a *principle filter* if there exists some $x \in B$ such that $F = \{y \in B \,|x \leq y\}$.

(5) F is said to be an *ultrafilter* if for every $x \in B$, $x \in F$ or $\neg x \in F$.

Question 1.
Let B a Boolean algebra.

(1) Show that $<$ is an ordering with maximum 1 and minimum 0. Show that $f : B \to B'$ is an isomorphism of Boolean Algebras iff f is an isomorphism between $(B, <)$ and $(B', <)$.
Solution: The first clause is routine check. The second: it is enough to see that $\wedge, \vee, \neg, 0, 1$ are all defined using $<$. For instance, $x \wedge y = \min\{x, y\}$ and $\neg x$ is the unique x' such that $x' \wedge x = 0$ and $x' \vee x = 1$.

(2) Let $S(B)$ be the set of all ultrafilters on B. Show that every filter on B can be extended to an ultrafilter on B, and thus $S(B)$ is not empty.
This is a Boolean algebra (of subsets of B), and it is called the *Stone space* of B.
Solution: the same way in which we show that every filter on sets is extended to an ultrafilter (Zorn's lemma).

(3) For $x \in B$, let $[x] = \{y \in S(B) \,|x \in y\}$. Check that $\{[x] \,|x \in B\}$ defines a closed-open basis for a topology on $S(B)$.
Solution: $[x] \cap [y] = [x \wedge y]$, so it is a basis, and $S(B) \setminus [x] = [\neg x]$ so it is closed.

(4) Show that $S(B)$ is compact Hausdorff space.
Solution: Hausdorff – if $u_1 \neq u_2 \in B$ then there is some $x \in u_1$, $x \notin u_2$. So $u_1 \in [x]$ and $u_2 \in [\neg x]$. Compact: assume $\{[x_i] \,|i \in I\}$ cover $S(B)$. If $[x_{i_1}], \ldots, [x_{i_n}]$ do not cover $S(B)$, then there is an ultrafilter D

containing $\neg x_{i_1}, \ldots, \neg x_{i_n}$. In particular $\neg x_{i_1} \wedge \ldots \wedge \neg x_{i_n} \neq 0$. It follows that $\{\neg x_i \,| i \in I\}$ generates a filter. But this is a contradiction to assumption.

(5) Check that every closed-open set in $S(B)$ is of the form $[x]$ for some $x \in B$. Let B' be the set of all closed-open sets in $S(B)$.
Solution: Suppose $u \subseteq S(B)$ is closed-open. Let $U = \{x \,| [x] \subseteq u\}$. Since u is open, U covers u. Since u is closed, it is compact. So there is some finite $U_0 \subseteq U$ which covers U. Let $x_u = \bigvee_{x \in U_0} x$. Then $u = [x_u]$.

(6) Let $f : B \to B'$ be $f(x) = [x]$. Show that f is an isomorphism of Boolean Algebras.
This is *Stone's representation theorem.*
Solution: if $x \neq y$ then wlog $x \wedge y \neq x$. Then $x \wedge (\neg y) \neq 0$ and any ultrafilter containing $x \wedge (\neg y)$ does not contain y. Now it is enough to see that f preserves the ordering, which is clear.

(7) Deduce that B can is isomorphic to a sub-Boolean algebra of the power set $\mathcal{P}(X)$ for some set X.
Solution: the set of all closed-open subsets of $S(B)$ is a sub algebra of $\mathcal{P}(S(B))$.

Question 2.
Let L be a signature. Let B be *Lindenbaum–Tarski* algebra of L be the set equivalence classes (under elementary equivalence) of sentences of L.

(1) We give B a structure of a Boolean Algebra in the obvious way (for instance, given $[\varphi], [\psi]$, we define $[\varphi] \wedge [\psi] = [\varphi \wedge \psi]$). Make sure it is well defined.
(2) For a complete theory T, let $F(T) := \{[\varphi] \,| \varphi \in T\}$. Prove that $F(T)$ is in $S(B)$.
Solution: Note that if $[\varphi] \in u = F(T)$ then $\varphi \in T$. So if $[\bot] = 0 \in u$, then T is not consistent. If $[\varphi] \leq [\psi]$ and $[\varphi] \in u$, then $\varphi \models \psi$, so $\psi \in T$, so u is closed upwards. If $[\varphi], [\psi] \in u$, then $[\varphi \wedge \psi] \in T$, and so u is a filter.
(3) Prove that the following are equivalent:
   (a) F is an isomorphism.
   (b) The finiteness / compactness theorem.

Solution: (b) to (a): Easily F is injective. F is onto, because if $u \in S(B)$, then consider $T := \{\varphi \,| [\varphi] \in u\}$. If we show that T is a complete theory, we're done. First T is consistent: if $\varphi_1, \ldots, \varphi_n \in T$ then $[\varphi_1], \ldots, [\varphi_n] \in u$, so $[\varphi_1 \wedge \ldots \wedge \varphi_n] \in u$ and hence $[\varphi_1 \wedge \ldots \wedge \varphi_n] \neq 0 = [\bot]$. So $\bot$ is not logically equivalent with $\varphi_1 \wedge \ldots \wedge \varphi_n$, so it has a model. By compactness, T is consistent. T is complete because u is an ultrafilter. T is closed under implication follows from completeness + consistency.
(a) to (b): If Σ is an inconsistent set of sentences. Then by (a), every $u \in S(B)$ contains some $[\neg\varphi]$ for some $\varphi \in \Sigma$ (else Σ is consistent). By the compactness of $S(B)$, it follows that there is a finite Σ_0 such that every $u \in S(B)$ contains some $[\neg\varphi]$ for $\varphi \in \Sigma_0$. But then Σ_0 is not consistent (else we could take the theory of a model of Σ_0, and by (2) this gives a contradiction).

Question 3.
Recall exercise 2. All the axioms of Boolean algebra are equational, and hence by that Question 3, (3) there, substructures, homomorphic images and products of Boolean algebras are also Boolean algebras. Let 2 be the trivial Boolean algebra: the universe is just $\{0, 1\}$ and everything is defined by the axioms.

(1) Let X be a set. Show that the algebra on $\mathcal{P}(X)$ is isomorphic to 2^X (i.e. the product $\prod \langle 2_x | x \in X \rangle$ where $2_x = 2$ for all $x \in X$).
Solution: the isomorphism is u maps to the characteristic function of u. It is easy to check that this is an isomorphism (one can check that it preservers the ordering).

(2) Show that the theory of Boolean algebras (i.e. the set of axioms in the definitions) is elementarily equivalent to $\Sigma := \{\varphi | 2 \models \varphi, \varphi \text{ is equational}\}$. (See Ex. 2).
Hint: use Exercise 2, Question 3 (3).
Solution: Obviously, the set of axioms in the definition are equational and so it is a subset of Σ. On the other hand, if B is a Boolean algebra, then B can be embedded in $\mathcal{P}(X)$ for some set X. But $\mathcal{P}(X) \cong 2^X$, so B can be embedded in 2^X, which is a model of Σ by Exercise 2, Question 3 (3), so it is also a model of Σ by that Question.