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17.09.2013

Summerschool Reduced Basis Methods 2013
Exercise Sheet

Part A: POD for dynamical systems

POD Galerkin ansatz. Consider the linear parabolic differential equation

$$M\dot{y}(t) + Ay(t) = f(t) + Bu(t), \qquad My(0) = y_\circ.$$

1. Find a linear and bounded operator $S : U \to Y$ and $\hat{y} \in Y$ such that $y = Su + \hat{y}$.
2. Complete line 7 and implement the function `Rom` in the algorithm

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Algorithm 1 (PodGalerkin)

Require: $M, A, f, y_\circ, B, u, \Delta t, \ell_{\max}$.

1: Solve $\hat{y} = \texttt{State}(M, A, f, y_\circ, \Delta t)$
2: Solve $y_1 = \texttt{State}(M, A, Bu, 0, \Delta t)$ and $y_2 = \texttt{State}(M, A, f + Bu, y_\circ, \Delta t)$
3: Solve $\Psi_1 = \texttt{Pod}(\Delta t, M, y_1, \ell_{\max})$ and $\Psi_2 = \texttt{Pod}(\Delta t, M, y_2, \ell_{\max})$
4: **for** $\ell = 1, ..., \ell_{\max}$ **do**
5: Determine $[M_1^\ell, A_1^\ell, B_1^\ell] = \texttt{Rom}(\Psi_1^\ell, M, A, B)$ and $[M_2^\ell, A_2^\ell, B_2^\ell, f_2^\ell, y_{\circ 2}^\ell] = \texttt{Rom}(\Psi_2^\ell, M, A, B, f, y_\circ)$
6: Compute $x_1^\ell = \texttt{State}(M_1^\ell, A_1^\ell, B_1^\ell u, 0, \Delta t)$ and $x_2^\ell = \texttt{State}(M_2^\ell, A_2^\ell, f_2^\ell + B_2^\ell u, y_{\circ 2}^\ell, \Delta t)$
7: Determine the corresponding high-dimensional states $y_1^\ell = ...,\ y_2^\ell = ...$
8: Compute $e_1^\ell = \|y_2 - y_1^\ell\|_Y$ and $e_2^\ell = \|y_2 - y_2^\ell\|_Y$
9: **end for**

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3. Visualize e_1, e_2 and the first few Pod elements of Ψ_1, Ψ_2.

Part B: Optimal control of PDEs

Optimization problem. We consider the pde-constrained optimal control problem

$$\min J(y,u) = \frac{1}{2}\int_0^T \|y(t) - \vec{y}_Q(t)\|_H^2 \mathrm{d}t + \frac{1}{2}\|y(T) - \vec{y}_\Omega\|_H^2 + \frac{\sigma}{2}\|u\|_U^2$$

subject to

$$M\dot{y}(t) + Ay(t) = f(t) + Bu(t) \ \&\ My(0) = y_\circ, \qquad u_a(t) \le u(t) \le u_b(t).$$

Optimality system. An optimal control-state pair $(\bar{y}, \bar{u}) \in Y \times U$ is given by

$$\begin{aligned} M\dot{y}(t) + Ay(t) - f(t) - Bu(t) &= 0, & y(0) &= y_\circ \\ -M\dot{p}(t) + Ap(t) + My(t) - y_Q(t) &= 0, & Mp(T) &= y_\Omega - My(T) \\ u(t) - \mathbb{P}(\sigma^{-1}B^\star p(t)) &= 0 \end{aligned}$$

where $\mathbb{P}(u) = \min(\max(u, u_a), u_b)$ is the canonical projection of U onto $[u_a, u_b]$ and $y_Q(t) = M\vec{y}_Q(t)$, $y_\Omega = M\vec{y}_\Omega$.

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4. Find a linear and bounded operator $\tilde{S} : U \to Y$ and $\hat{p} \in Y$ such that $p = \tilde{S}u + \hat{p}$.
5. Define a selfmapping F on U such that the optimal control $\bar{u}$ is a fixpoint of F.
6. Formulate a condition of the regularisation parameter σ such that the corresponding Banach fixpoint iteration admits a unique solution.

Part C: Reduced order modellung for optimization problems

Optimization algorithm. We provide the following fixpoint strategy:

Algorithm 2 (SolverOptimizationProblem)

Require: initial control $u_\circ$, desired exactness ε, maximal iterations $k_{\max}$, inhomogeneous component $B^\star \hat{p}$

Set $k = 0$, $u = u_\circ$
repeat
  Compute $y_h = Su = \texttt{State}(M, A, Bu, 0, \Delta t)$
  Compute $p_h = \tilde{S}u = \texttt{fliplr}(\texttt{State}(M, A, -\texttt{fliplr}(My_h), -My(T), \Delta t))$
  Evaluate $u_+ = \mathbb{P}(\sigma^{-1}(B^\star p_h + B^\star \hat{p}))$
until $\|u_+ - u\|_U < \varepsilon$ **or** $k = k_{\max}$.
Set $u = u_+$ and $k = k + 1$
Return optimal control u.

7. Design an algorithm which combines the model reduction via POD with the provided optimization strategy.
8. Visualize the errors between the suboptimal controls u^ℓ and the optimal control u for $\ell = 1, ..., 15$.