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17.09.2013

Summerschool Reduced Basis Methods 2013
Solution Sheet

PART A: POD FOR DYNAMICAL SYSTEMS

POD Galerkin ansatz. Consider the linear parabolic differential equation

$$M\dot{y}(t) + Ay(t) = f(t) + Bu(t), \qquad My(0) = y_\circ.$$

1. Find a linear and bounded operator $S : U \to Y$ and $\hat{y} \in Y$ such that $y = Su + \hat{y}$.
2. Complete line 7 and implement the function `Rom` in the algorithm

Algorithm 1 (PodGalerkin)

Require: $M, A, f, y_\circ, B, u, \Delta t, \ell_{\max}$.
1: Solve $\hat{y} = \texttt{State}(M, A, f, y_\circ, \Delta t)$
2: Solve $y_1 = \texttt{State}(M, A, Bu, 0, \Delta t)$ and $y_2 = \texttt{State}(M, A, f + Bu, y_\circ, \Delta t)$
3: Solve $\Psi_1 = \texttt{Pod}(\Delta t, M, y_1, \ell_{\max})$ and $\Psi_2 = \texttt{Pod}(\Delta t, M, y_2, \ell_{\max})$
4: **for** $\ell = 1, ..., \ell_{\max}$ **do**
5: Determine $[M_1^\ell, A_1^\ell, B_1^\ell] = \texttt{Rom}(\Psi_1^\ell, M, A, B)$ and $[M_2^\ell, A_2^\ell, B_2^\ell, f_2^\ell, y_{\circ 2}^\ell] = \texttt{Rom}(\Psi_2^\ell, M, A, B, f, y_\circ)$
6: Compute $x_1^\ell = \texttt{State}(M_1^\ell, A_1^\ell, B_1^\ell u, 0, \Delta t)$ and $x_2^\ell = \texttt{State}(M_2^\ell, A_2^\ell, f_2^\ell + B_2^\ell u, y_{\circ 2}^\ell, \Delta t)$
7: Determine the corresponding high-dimensional states $y_1^\ell = ...,\ y_2^\ell = ...$
8: Compute $e_1^\ell = \|y_2 - y_1^\ell\|_Y$ and $e_2^\ell = \|y_2 - y_2^\ell\|_Y$
9: **end for**

3. Visualize e_1, e_2 and the first few Pod elements of Ψ_1, Ψ_2.

Solutions.

1. We consider the homogeneous controlled and the inhomogeneous uncontrolled pdes

$$M\dot{y}(t) + Ay(t) = Bu(t) \ \&\ My(0) = 0, \qquad M\dot{y}(t) + Ay(t) = f(t) \ \&\ My(0) = y_\circ,$$

then S is the linear, bounded solution operator to the first problem and $\hat{y}$ is the solution to the second one.

2. Let N be the dimension of the high-dimensional system and n be the number of control components, i.e. $M, A \in \mathbb{R}^{N\times N}$ and $B \in \mathbb{R}^{N\times n}$. The function `Rom` assembles the system matrices of the reduced order model. For this purpose, the high-dimensional pde is transformed onto the space spanned by the POD basis elements which is provided by multiplication from the left with $(\Psi^\ell)^{\mathrm{T}} \in \mathbb{R}^{\ell\times N}$; further, the solution vector $y(t) \in \mathbb{R}^N$ is replaced by its expansion in the POD basis, $\sum x_i(t)\psi_i^\ell = \Psi^\ell \cdot x(t)$ where $x(t) \in \mathbb{R}^\ell$. The reduced pde is

$$M^\ell \dot{x}(t) + A^\ell x(t) = f^\ell(t) + B^\ell u(t), \qquad M^\ell x(0) = y_\circ^\ell$$

where $M^\ell, A^\ell, B^\ell, f^\ell, y_\circ^\ell$ are calculated by the function `ROM` which is an implementation of

Algorithm 2 (ReducedOrderModel)

Require: $\Psi^\ell, M, A, B, f, y_\circ, y_Q, y_\Omega$.
$M^\ell = (\Psi^\ell)^{\mathrm{T}} M \Psi^\ell \in \mathbb{R}^{\ell\times\ell}$, $A^\ell = (\Psi^\ell)^{\mathrm{T}} A \Psi^\ell \in \mathbb{R}^{\ell\times\ell}$, $B^\ell = (\Psi^\ell)^{\mathrm{T}} B \in \mathbb{R}^{\ell\times n}$.
if `nargout` ≥ 5 **then**
$f^\ell(t) = (\Psi^\ell)^{\mathrm{T}} f(t) \in \mathbb{R}^\ell$, $y_\circ^\ell = (\Psi^\ell)^{\mathrm{T}} y_\circ \in \mathbb{R}^\ell$
end if
if `nargout` ≥ 7 **then**
$y_Q^\ell(t) = (\Psi^\ell)^{\mathrm{T}} y_Q(t) \in \mathbb{R}^\ell$, $y_\Omega^\ell = (\Psi^\ell)^{\mathrm{T}} y_\Omega \in \mathbb{R}^\ell$.
end if

Notice: The coefficients for y_Q, y_Ω are required later.

The full solutions $y_1^\ell(t), y_2^\ell(t) \in \mathbb{R}^N$ are reconstructed by $y_1^\ell(t) = \Psi_1^\ell \cdot x_1(t) + \hat{y}(t)$ and $y_2^\ell(t) = \Psi_2^\ell \cdot x_2(t)$.

3. We have a look at the inhomogeneous, controlled state solution $y_2 = y_1 + \hat{y}$, at the errors between the high-dimensional FE model and the low-dimensional POD model and at the shape of the POD basis elements of the two ansatzes.

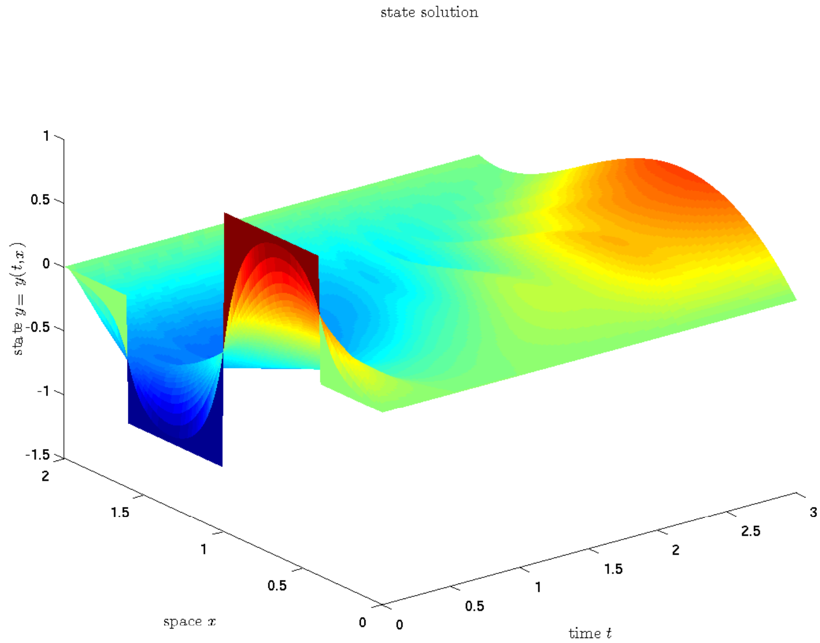

Fig. 1: The state solution with discontinuous initialization. As typical for linear parabolic equations, the initial value is smoothend imediately.

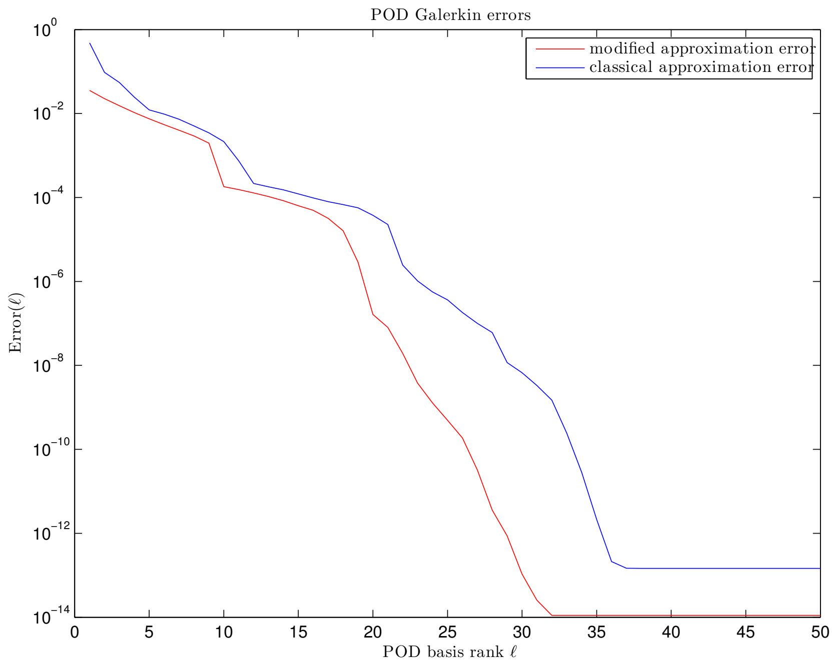

Fig. 2: The state errors caused by the model reduction. As one sees, the modified ansatz where just the snapshots of the homogeneous solution component $y_1 = Su$ are used for the POD basis creation leads to significantly better results than the classical ansatz where the snapshots of $y_2 = Su + \hat{y}$ generate the POD basis.

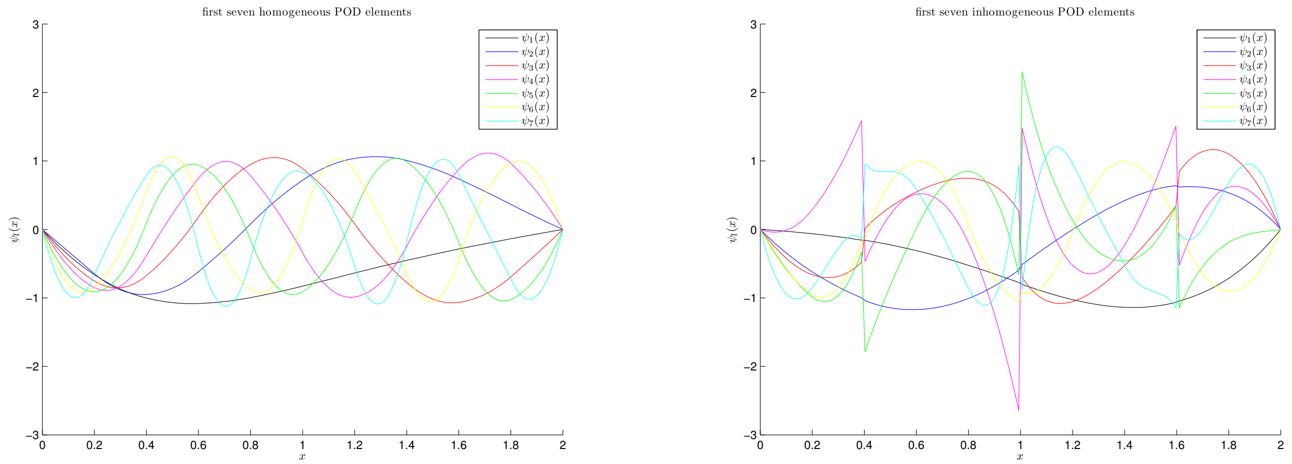

Fig. 3: The first POD basis elements ψ_1^l of the homogeneous ansatz (left) and ψ_2^l of the inhomogeneous one (right). As one sees, the rows ψ_2^l of Ψ_2 include jumps at the positions where $y_\circ$ is discontinuous to be able to reconstruct the initial value which is not required for the homogeneous ansatz.

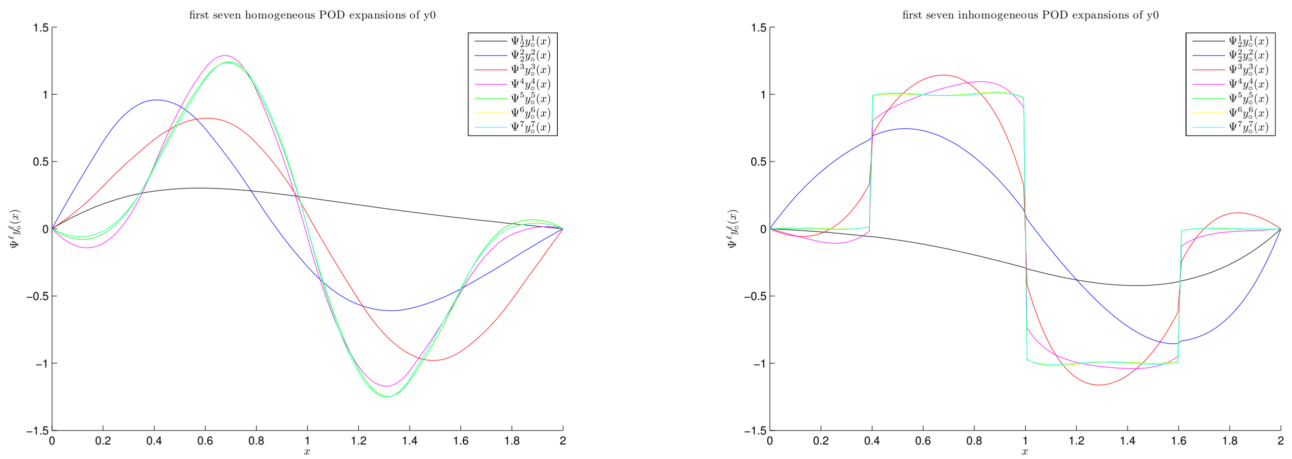

Fig. 4: The reconstruction of the initial value, $\Psi^\ell y_0^\ell$ for different POD basis ranks ℓ with the modified ansatz (left) and the classical one (right). The approximation with the inhomogeneous basis Ψ_2 works quite well where with Ψ_1, it is neither required nor possible to build up $y_\circ$ accurately.

Part B: Optimal control of PDEs

Optimization problem. We consider the pde-constrained optimal control problem

$$\min J(y,u) = \frac{1}{2}\int_0^T \|y(t) - \vec{y}_Q(t)\|_H^2 \,\mathrm{d}t + \frac{1}{2}\|y(T) - \vec{y}_\Omega\|_H^2 + \frac{\sigma}{2}\|u\|_U^2$$

subject to

$$M\dot{y}(t) + Ay(t) = f(t) + Bu(t) \;\&\; My(0) = y_\circ, \qquad u_a(t) \le u(t) \le u_b(t).$$

Optimality system. An optimal control-state pair $(\bar{y}, \bar{u}) \in Y \times U$ is given by

$$\begin{aligned} M\dot{y}(t) + Ay(t) - f(t) - Bu(t) &= 0, & My(0) &= y_\circ, \\ -M\dot{p}(t) + Ap(t) + My(t) - y_Q(t) &= 0, & Mp(T) + My(T) - y_\Omega, &= 0 \\ u(t) - \mathbb{P}(\sigma^{-1}B^\star p(t)) &= 0 \end{aligned}$$

where $\mathbb{P}(u) = \min(\max(u, u_a), u_b)$ is the canonical projection of U onto $[u_a, u_b]$ and $y_Q(t) = M\vec{y}_Q(t)$, $y_\Omega = M\vec{y}_\Omega$.

Lagrange calculus. The optimality system is derived by differentiating the Lagrange function

$$L(y,u,p) = \hat{J}(y,u) + \int_0^T \langle E(y,u)(t), p(t)\rangle_{L^2(V',V} \,\mathrm{d}t$$

with the modified objective functional $\hat{J} : Y \times U \to \mathbb{R}$ and the constraints operator $E : Y \times U \to \mathcal{L}^2(\Theta, \mathbb{R}^N)$,

$$\hat{J}(y,u) = \int_0^T \|y(T) - (\vec{y}_Q(t) - \hat{y}(t))\|_H^2 \,\mathrm{d}t + \|y(t) - (\vec{y}_\Omega(t) - \hat{y}(T))\|_H^2 + \frac{\sigma}{2}\|u\|_U^2,$$
$$E(y,u) = M\dot{y}(t) + Ay(t) - Bu(t).$$

Notice: The resulting variational inequality $\forall \tilde{u} \in [u_a, u_b] : \langle \sigma u - B^\star p, \tilde{u} - u\rangle_U \geq 0$ is equivalent to $u = \mathbb{P}(\sigma^{-1} B^\star p)$.

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4. Find a linear and bounded operator $\tilde{S} : U \to Y$ and $\hat{p} \in Y$ such that $p = \tilde{S}u + \hat{p}$.
5. Define a selfmapping F on U such that the optimal control $\bar{u}$ is a fixpoint of F.
6. Formulate a condition of the regularisation parameter σ such that the corresponding Banach fixpoint iteration admits a unique solution.

Solutions.

4. Consider the two initial value problems

$$\begin{aligned} -M\dot{p}(t) + Ap(t) &= -MSu(t) \qquad \& \qquad Mp(T) = -MSu(T), \\ -M\dot{p}(t) + Ap(t) &= y_Q(t) - M\hat{y}(t) \qquad \& \qquad Mp(T) = y_\Omega - M\hat{y}(T), \end{aligned}$$

then $\tilde{S}$ is the linear, bounded solution operator to the first one and $\hat{p}$ is the solution of the second one.

5. Choose $F(u) = \mathbb{P}(\sigma^{-1}(B^\star \tilde{S}u + B^\star \hat{p}))$, then (y,u,p) solves the optimality system if and only if $y = Su + \hat{y}$, $p = \tilde{S}u + \hat{p}$ and $F(u) = u$.

6. We show that F is a contraction for suitable σ, i.e. that

$$\exists C \in (0,1) : \forall u, \tilde{u} \in U : \|F(u) - F(\tilde{u})\|_U \leq C\|u - \tilde{u}\|_U.$$

Since $\mathbb{P}$ is an orthogonal projector on the nonempty, closed, convex set $[u_a, u_b]$, $\mathbb{P}$ is Lipschitz continuous of order 1, i.e. $\|\mathbb{P}(u) - \mathbb{P}(\tilde{u})\|_U \leq 1 \cdot \|u - \tilde{u}\|_U$ for all $u, \tilde{u} \in U$. Further, the operator $B^\star \tilde{S}$ is bounded. Hence:

$$\|Fu - F\tilde{u}\|_U \leq \sigma^{-1}\|B^\star \tilde{S}\| \, \|u - \tilde{u}\|.$$

Choosing $C = \sigma^{-1}\|B^\star \tilde{S}\|$, we get $C \in (0,1)$ for $\sigma > \|B^\star \tilde{S}\|$.

Part C: Reduced order modellung for optimization problems

Optimization algorithm. We provide the following fixpoint strategy:

Algorithm 3 (SolverOptimizationProblem)

Require: initial control $u_\circ$, desired exactness ε, maximal iterations $k_{\max}$, inhomogeneous component $B^\star \hat{p}$

Set $k = 0$, $u = u_\circ$
repeat
  Compute $y_h = Su = \texttt{State}(M, A, Bu, 0, \Delta t)$
  Compute $p_h = \tilde{S}u = \texttt{fliplr}(\texttt{State}(M, A, -\texttt{fliplr}(My_h), -My(T), \Delta t))$
  Evaluate $u_+ = \mathbb{P}(\sigma^{-1}(B^\star p_h + B^\star \hat{p}))$
until $\|u_+ - u\|_U < \varepsilon$ **or** $k = k_{\max}$.
Set $u = u_+$ and $k = k + 1$
Return optimal control u.

Notice that the adjoint equation can be solved with the forward routine `State` as well by backwards transformation in time:

$$M\dot{q}(t) + Aq(t) = -M(Su)(T-t) \;\&\; Mq(0) = -M(Su)(T), \qquad p(t) = q(T-t).$$

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7. Design an algorithm which combines the model reduction via POD with the provided optimization strategy.
8. Visualize the errors between the suboptimal controls u^ℓ and the optimal control u for $\ell = 1, \dots, 15$.

Solutions.

7. We have to calculate the inhomogeneous component $B^\star \hat{p}$, to solve the high-dimensional state equation to build up the snapshot matrix, to construct the reduced order system matrices and to execute the optimizer with the reduced objects as input parameters:

Algorithm 4 (ReducedOrderOptimization)

Calculate $\hat{y}(t) = \texttt{State}(M, A, f, y_\circ, \Delta t)$.
Calculate $\hat{p}(t) = \texttt{fliplr}(\texttt{State}(M, A, \texttt{fliplr}(y_Q - M\hat{y}), y_\Omega - M\hat{y}(T), \Delta t)) \in \mathbb{R}^N$.
Construct inhomogeneous component $(B^\star \hat{p})(t) \in \mathbb{R}^n$.
Execute optimizer $\bar{u}(t) = \texttt{Solver}(u_\circ, \epsilon, k_{\max}, M, A, B, \Delta t, \sigma, u_a, u_b, B^\star \hat{p}) \in \mathbb{R}^n$.
Calculate snapshots $y(t) = \texttt{State}(M, A, Bu_\circ, 0, \Delta t) \in \mathbb{R}^N$.
Determine a rank-$\ell_{\max}$ POD basis $\Psi = \texttt{Pod}(\Delta t, M, y, \ell_{\max}) \in \mathbb{R}^{N \times \ell_{\max}}$.
for $\ell = 1, \dots, \ell_{\max}$ **do**
    Assemble reduced order model $[M^\ell, A^\ell, B^\ell] = \texttt{Rom}(\Psi^\ell, M, A, B) \in \mathbb{R}^{\ell \times \ell} \times \mathbb{R}^{\ell \times \ell} \times \mathbb{R}^{\ell \times n}$.
    Execute optimizer $u^\ell(t) = \texttt{Solver}(u_\circ, \epsilon, k_{\max}, M^\ell, A^\ell, B^\ell, \Delta t, \sigma, u_a, u_b, B^\star \hat{p}) \in \mathbb{R}^n$.
    Compute control error $e(\ell) = \|\bar{u} - u^\ell\|_U$.
end for

8. We visualize the optimal control functions $\bar{u}_i$ first.

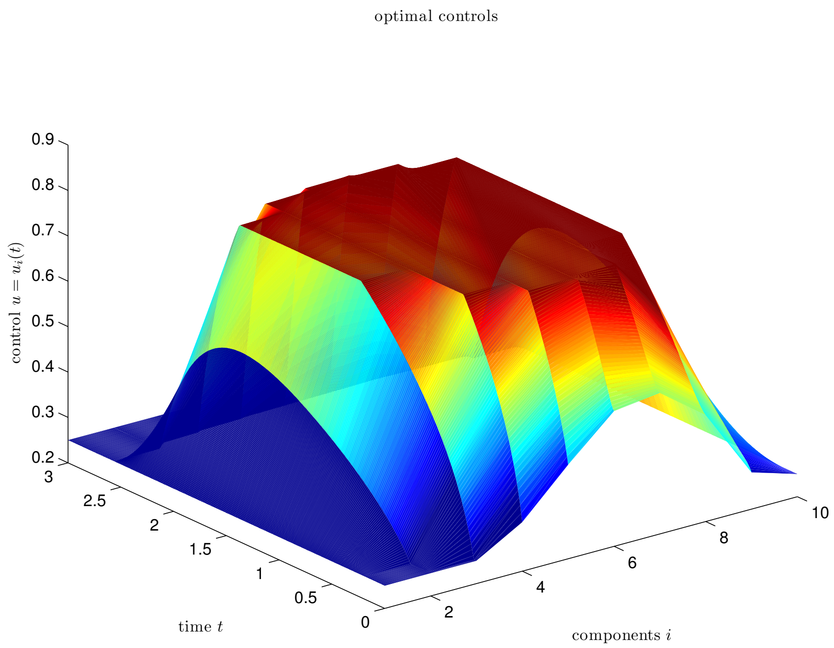

Fig. 5: The control bounds $u_a = 0.25$ and $u_b = 0.75$ are active for u_1, u_{10} and for the central components.

An efficient decay of the control errors can just be expected if the initial control guess $u_\circ$ is already close to the optimal control $\bar{u}$. If this is not the case, the procedure may by repeated several times to construct a sequence $(u_k^\ell)_k$ with initialization $u_{\circ 0}^\ell = u_\circ$ and $u_{\circ k}^\ell = u_{k-1}^\ell$, $k = 2, \dots$.

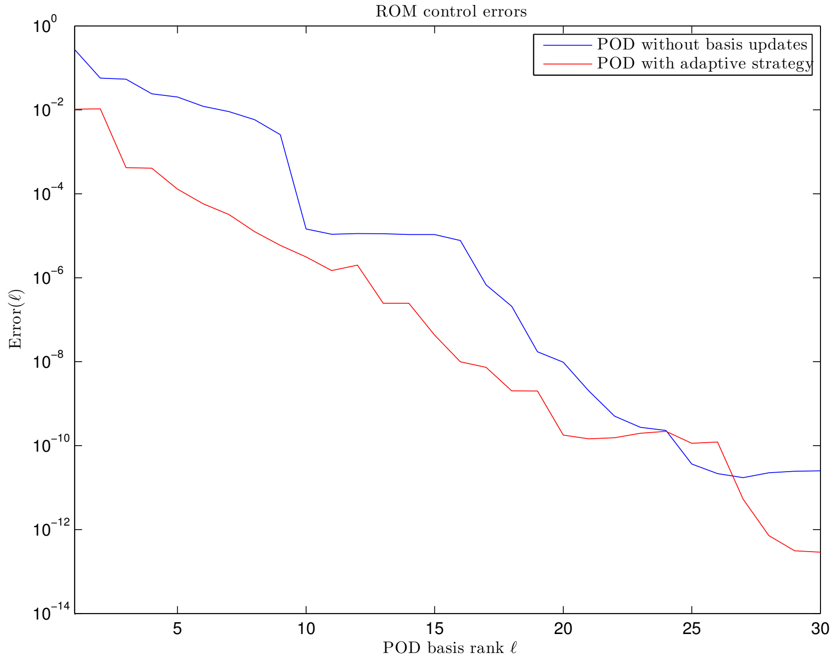

Fig. 6: The control errors with respect to the reduced control solutions u_1^ℓ, calculated without a basis adaptivity strategy, and u_2^ℓ where the POD basis is updated once by initialiting the snapshots with u_1^ℓ. Indeed, the snapshots to u_1^ℓ already numerically coincide with those to $\bar{u}$, i.e. a second basis update leads to no improvement in the error decay.

Further information, especially concerning the asymptotic behavior of the errors, decay rates, a priori error bounds and a posteriori error estimators, can be found in the literature.

Literature

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