Tutorial *Analysis I*

Return: 12:00, Friday 14.12.

Exercise 21:

In the definition of Cauchy sequences (s_n), it is important that for given $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that the estimate $|s_n - s_m| < \epsilon$ holds **for all** $n, m > N$. Show:

(i) If (s_n) is a bounded sequence with the property: for every $\epsilon > 0$ there exists N such that $|s_{n+1} - s_n| < \epsilon$ for all $n > N$, then (s_n) need not converge.

(ii) A bounded sequence (s_n) need not converge even if $|s_{n+1} - s_n| < 1/n$ for all $n > N$ with a suitable $N \in \mathbb{N}$.

(iii) However, if (s_n) has the property that $|s_{n+1} - s_n| < 2^{-n}$ for all $n > N$ and a suitable $N \in \mathbb{N}$, then (s_n) converges.

Exercise 22:

Find an estimate S_N for the partial sum $s_N = \sum_{n=1}^{N} \frac{1}{n}$ and determine $\Delta_N \geq 0$ (as small as possible) with the property $|S_N - s_N| \leq \Delta_N$. Then calculate your estimated value for N from the following list:

(i) Hamburgers sold by McDonalds: $2 \cdot 10^9$.

(ii) Number of ants on earth: 10^{15}

(iii) Cups of water in al oceans: $6 \cdot 10^{21}$

(iv) Atoms in the atmosphere: $200 \cdot 10^{42}$

(v) Atoms in the universe: 10^{78}

Exercise 23:

Let (a_n) be a sequence with $a_n \geq 0$ for all n and let $p > 1$. Prove that $\sum a_n^p$ converges if $\sum a_n$ converges.

Exercise 24:

Show that $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$ with the trick $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$ and test for convergence/divergence:

$$\sum_{n=1}^{\infty} \frac{n-1}{n^2}, \qquad \sum_{n=1}^{\infty} \frac{1}{n^n}, \qquad \sum_{n=1}^{\infty} \frac{n!}{n^n}, \qquad \sum_{n=1}^{\infty} (\sqrt{n+1} - \sqrt{n}), \qquad \sum_{n=1}^{\infty} \frac{1}{\sqrt{n!}}$$