Universität Konstanz
Fachbereich Mathematik und Statistik
Prof. Dr. Salma Kuhlmann
Dr. Annalisa Conversano
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ÜBUNGEN ZUR VORLESUNG POSITIVE POLYNOME

BLATT 02

These exercises will be collected Tuesday 4 May in the mailbox n.14 of the Mathematics department.

1. Let A be a commutative ring with 1. For an ordering $P \subseteq A$ let

$$\mathrm{F_P} := \mathrm{ff}(A/\mathfrak{p})$$

be the field of fractions of $A/\mathfrak{p}$, where $\mathfrak{p} := -P \cap P$. For every $a \in A$ we denote by $\bar{a}$ the equivalence class of a in $A/\mathfrak{p}$. Define

$$\forall\, a, b \in A,\ b \notin \mathfrak{p} \qquad \frac{\bar{a}}{\bar{b}} \geqslant_P 0 \;\Leftrightarrow\; ab \in P.$$

Show that:

(*a*) $\geqslant_P$ is well-defined on $\mathrm{F_P}$,

(*b*) $(\mathrm{F_P}\,, \geqslant_P)$ is an ordered field.

2. Let $n \in \mathbb{N}$. Let K be a field, $V \subseteq K^n$ an algebraic subset, $I \subseteq K[\mathrm{x}_1, \ldots, \mathrm{x}_n]$ an ideal.

(I) Show that:

(*a*) $\mathcal{I}(V)$ is an ideal,

(*b*) $\mathcal{Z}(\mathcal{I}(V)) = V$,

(*c*) the map $V \mapsto \mathcal{I}(V)$ is an injection from the set of algebraic subsets of K^n into the set of ideals of $K[\mathrm{x}_1, \ldots, \mathrm{x}_n]$.

(II) Give an example where

$$I \subsetneq \mathcal{I}(\mathcal{Z}(I)).$$

3. Let A be a commutative ring with 1 such that $1+1 \in A^*$ and $M \subseteq A$ a quadratic module of A. Show that:

(a) $-M \cap M$ is an ideal of A;

(b) the following are equivalent:

(i) $a \in \sqrt{-M \cap M} := \{a \in A : \exists\, m \in \mathbb{N} \text{ s.t. } a^m \in -M \cap M\}$;

(ii) $a^{2m} \in -M \cap M$ for some $m \in \mathbb{N}$;

(iii) $-a^{2m} \in M$ for some $m \in \mathbb{N}$.

4. Let A be a commutative ring with 1. Show that if M is the quadratic module (resp., preordering) of A generated by $\{g_1, \dots, g_s\}$ and I is the ideal of A generated by $\{h_1, \dots, h_t\}$, then

$$M + I := \{g + h : g \in M,\, h \in I\}$$

is the quadratic module (resp., preordering) of A generated by $\{g_1, \dots, g_s, h_1, -h_1, \dots, h_t, -h_t\}$.

5. Let A be a commutative ring with 1 and $I \subseteq A$ an ideal. We recall that

- I is **prime** if $ab \in I \Rightarrow a \in I$ or $b \in I$.
- I is **radical** if $I = \sqrt{I} := \{a \in A : \exists\, m \in \mathbb{N} \text{ s.t. } a^m \in I\}$.
- I is **real** if $I = \sqrt[R]{I} := \{a \in A : \exists\, m \in \mathbb{N}\, \exists\, \sigma \in \sum A^2 \text{ s.t. } a^{2m} + \sigma \in I\}$.

(a) Show that any prime ideal is radical.

(b) Give an example of an ideal $I \subset K[\mathrm{x}_1, \dots, \mathrm{x}_n]$ (for some field K and $n \in \mathbb{N}$) which is radical and it is not prime.

(c) Give an example of an ideal $I \subset K[\mathrm{x}_1, \dots, \mathrm{x}_n]$ (for some field K and $n \in \mathbb{N}$) which is prime and it is not real.