

REAL ALGEBRAIC GEOMETRY LECTURE NOTES
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CONTENTS

1. Preorderings and positive cones	1
2. A crucial Lemma	3
3. Several consequences	3

1. PREORDERINGS AND POSITIVE CONES

Definition 1.1. (*Präordnung*) Let K be a field and let $T \subseteq K$ such that

- (i) $T + T \subseteq T$,
- (ii) $TT \subseteq T$,
- (iii) $a^2 \in T$ for every $a \in K$.

(where $T + T := \{t_1 + t_2 : t_1, t_2 \in T\}$ and $TT := \{t_1 t_2 : t_1, t_2 \in T\}$).
Then T is said to be a **preordering** (or **cone**) of K .

Definition 1.2. (*echte Präordnung*) A preordering T of a field K is said to be **proper** if $-1 \notin T$.

Definition 1.3. (*Positivkegel*) A proper preordering T of a field K is said to be a **positive cone** if $-T \cup T = K$, where $-T := \{-t : t \in T\}$.

Proposition 1.4. Let (K, \leq) be an ordered field. Then the set

$$P := \{x \in K : x \geq 0\}$$

is a positive cone of K . Conversely, if P is a positive cone of a field K , then $\forall x, y \in K$

$$x \leq y \Leftrightarrow y - x \in P$$

defines an ordering on K such that (K, \leq) is an ordered field.

Therefore for every field K there is a bijection between the set of the orderings on K and the set of the positive cones of K .

Notation 1.5. Let K be a field. We denote by $\sum K^2$ the set

$$\{a_1^2 + \cdots + a_n^2 : n \in \mathbb{N}, a_i \in K, i = 1, \dots, n\}.$$

Exercise 1.6. Let K be a field. Then

- (1) $\sum K^2$ is a preordering of K .
- (2) $\sum K^2$ is the smallest preordering of K , i.e. if T is a preordering of K , then $\sum K^2 \subseteq T$.
- (3) If K is real then $-1 \notin \sum K^2$ (i.e. $\sum K^2$ is a proper preordering).
- (4) If K is algebraically closed then it is not real.
- (5) Let (K, P) be an ordered real field, F a field and

$$\varphi : F \longrightarrow K$$

an homomorphism of fields. Then $Q := \varphi^{-1}(P)$ is an ordering of F (Q is said to be the **pullback** of P).

- (6) If P, Q are positive cones of K with $P \subseteq Q$, then $P = Q$.
- (7) In particular, if $\sum K^2$ is a positive cone (or ordering: see 1.4) of K , then it is the unique ordering of K .

Remark 1.7. Let K be a field with $\text{char}(K) \neq 2$. If $T \subseteq K$ is a preordering which is not proper (i.e. $-1 \in T$), then $T = K$.

Proof. For every $x \in K$,

$$x = \left(\frac{x+1}{2}\right)^2 + (-1) \left(\frac{x-1}{2}\right)^2 \in T.$$

□

Remark 1.8. Let $\mathcal{T} = \{T_i : i \in I\}$ be a family of preorderings of a field K . Then

(i)

$$\bigcap_{i \in I} T_i$$

is a preordering of K .

(ii) if $\forall i, j \in I \exists k \in I$ such that $T_i \cup T_j \subseteq T_k$, then

$$\bigcup_{i \in I} T_i$$

is a preordering of K .

2. A CRUCIAL LEMMA

Lemma 2.1. *Let K be a field and T a proper preordering of K . If $a \in K$ and $a \notin T$, then*

$$T - aT = \{t_1 - at_2 : t_1, t_2 \in T\}$$

is a proper preordering of K .

Proof. Since $K^2 \subseteq T$, also $K^2 \subseteq T - aT$. Clearly $(T - aT) + (T - aT) \subseteq T - aT$. Moreover $\forall t_1, t_2, t_3, t_4 \in T$,

$$(t_1 - at_2)(t_3 - at_4) = t_1t_3 + a^2t_1t_4 - a(t_1t_4 + t_2t_3) \in T - aT,$$

therefore $(T - aT)(T - aT) \subseteq (T - aT)$ and $T - aT$ is a preordering of K .

If $(T - aT)$ is not proper, then $-1 = t_1 - at_2$ for some $t_1, t_2 \in T$ with $t_2 \neq 0$, since T is proper. Therefore

$$a = \frac{1}{t_2^2}(1 + t_1)t_2 \in T,$$

contradiction.

3. SEVERAL CONSEQUENCES

Corollary 3.1. *Every maximal proper preordering of a field K is an ordering (positive cone: see 1.4) of K .*

Corollary 3.2. *Every proper preordering of a field K is contained in an ordering of K .*

Proof. Let T be a proper preordering. Let

$$\mathcal{T} = \{T' : T' \supseteq T, T' \text{ is a proper preordering of } K\}.$$

\mathcal{T} is non-empty and for every ascending chain of \mathcal{T}

$$T_{i_1} \subseteq T_{i_2} \subseteq \dots \subseteq T_{i_k} \subseteq \dots$$

by 1.8(ii) $\bigcup T_{i_j}$ is a proper preordering containing T and Zorn's Lemma applies.

Let P be a maximal element of \mathcal{T} . Then P is a maximal preordering of K containing T , and by 3.1 P is an ordering. □

Corollary 3.3. *Let T be a proper preordering of a field K . Then*

$$T = \bigcap \{P : T \subseteq P, P \text{ positive cone of } K\}.$$

(\subseteq) It is obvious.

(\supseteq) Let $a \in K$ such that a is contained every positive cone containing T . If $a \notin T$, then by Lemma 2.1 $T - aT$ is a proper preordering of K . By Corollary 3.2, $T - aT$ is contained in a positive cone P of K . Then $-a \in P$ and $a \notin P$.

Corollary 3.4. (*Characterization of real fields*) Let K be a field. The following are equivalent:

- (1) K is real (i.e. K has an ordering).
- (2) K has a proper preordering.
- (3) $\sum K^2$ is a proper preordering (i.e. $-1 \notin \sum K^2$).
- (4) $\forall n \in \mathbb{N}, \forall a_1, \dots, a_n \in K$

$$\sum_{i=1}^n a_i^2 = 0 \Rightarrow a_1 = \dots = a_n = 0.$$

Proof. (1) \Rightarrow (2) \Rightarrow (3) obvious. We show now (3) \Leftrightarrow (4).

(\Rightarrow) Let $\sum_{i=1}^n a_i^2 = 0$ and suppose $a_i \neq 0$ for some $1 \leq i \leq n$. Say $a_n \neq 0$. Then

$$\frac{a_1^2 + \dots + a_n^2}{a_n^2} = 0,$$

and

$$\left(\frac{a_1}{a_n}\right)^2 + \dots + \left(\frac{a_{n-1}}{a_n}\right)^2 + 1 = 0.$$

Therefore $-1 \in \sum K^2$, contradiction.

(\Leftarrow) Suppose $-1 \in \sum K^2$, so

$$-1 = b_1^2 + \dots + b_s^2,$$

for some $s \in \mathbb{N}$ and $b_1, \dots, b_s \in K$. Then

$$1 + b_1^2 + \dots + b_s^2 = 0$$

and $1 = 0$, contradiction.

To complete the proof note that if $-1 \notin \sum K^2$ then $\sum K^2$ is a proper preordering, and by Corollary 3.2 K has an ordering. This proves (3) \Rightarrow (1). \square

Corollary 3.5. (*Artin*) Let K be a real field. Then

$$\sum K^2 = \bigcap \{P : P \text{ is an ordering of } K\}.$$

In other words, if K is a real field and $a \in K$, then

$$a \geq_P 0 \text{ for every ordering } P \Leftrightarrow a \in \sum K^2.$$