REAL ALGEBRAIC GEOMETRY LECTURE NOTES (05: 03/11/09)

SALMA KUHLMANN

Contents

1. Real closed fields

We first recall Artin-Schreir characterization of real closed fields:

Proposition 1.1. *(Artin-Schreir, 1926) Let K be a field. The following are equivalent:*

- (i) *K is real closed.*
- (ii) *K has an ordering P which does not extend to any proper algebraic extension.*
- (iii) *K is real, has no proper algebraic extension of odd degree, and*
$$K \;=\; K^2 \;\cup\; -(K^2).$$

Corollary 1.2. *If K is a real closed field then*
$$K^2 = \{a^2 : a \in K\}$$
is the unique ordering of K.

Proof. Since K is a real closed field, by (ii) it has an ordering P which does not extend to any proper algebraic extension.

Let $b \in P$. Then $b = a^2$ for some $a \in K$, otherwise P extends to an ordering of $K(\sqrt{b})$, which is a proper algebraic extension of K.

Therefore $P = K^2$. □

Remark 1.3. We denote by $\sum K^2$ the unique ordering of a real closed field K, even though we know that $\sum K^2 = K^2$, to avoid any confusion with the cartesian product $K \times K$.

Corollary 1.4. *Let $(K, \leqslant)$ be an ordered field. Then K is real closed if and only if*

- (a) *every positive element in K has a square root in K, and*
- (b) *every polynomial of odd degree has a root in K.*

Examples 1.5. $\mathbb{R}$ is real closed and $\mathbb{Q}$ is not.

2. The algebraic closure of a real closed field

Lemma 2.1. *(Hilfslemma) If K is a field such that K^2 is an ordering of K, then every element of $K(\sqrt{-1})$ is a square.*

Proof. Let $x = a + \sqrt{-1}\,b \in K(\sqrt{-1})$, $a, b \in K$, $b \neq 0$. We can suppose $b > 0$. We want to find $y \in K$ such that $x = y^2$.

K^2 is an ordering $\Rightarrow$ $a^2 + b^2 \in K^2$. Let $c \in K$, $c \geqslant 0$ such that

$$a^2 + b^2 = c^2.$$

Since $a^2 \leqslant a^2 + b^2 = c^2$, $|a| \leqslant c$, so $c + a \geqslant 0$, $c - a \geqslant 0$ $(-c \leqslant a \leqslant c)$.

Therefore $\frac{1}{2}(c \pm a) \in K^2$. Let $d, e \in K$, $d, e \geqslant 0$ such that

$$\frac{1}{2}(c+a) = d^2$$
$$\frac{1}{2}(c-a) = e^2.$$

So

$$d = \frac{\sqrt{c+a}}{\sqrt{2}} \qquad e = \frac{\sqrt{c-a}}{\sqrt{2}}$$

Now set $y := d + e\sqrt{-1}$. Then

$$\begin{aligned} y^2 &= (d + e\sqrt{-1})^2 \\ &= d^2 + (e\sqrt{-1})^2 + 2de\sqrt{-1} \\ &= \frac{1}{2}(c+a) - \frac{1}{2}(c-a) + 2\frac{1}{2}\sqrt{(c-a)(c+a)}\sqrt{-1} \\ &= \frac{1}{2}a + \frac{1}{2}a + \sqrt{c^2 - a^2}\sqrt{-1} \\ &= a + \sqrt{b^2}\sqrt{-1} \\ &= a + b\sqrt{-1} \\ &= x. \end{aligned}$$

□

Theorem 2.2. *(Fundamental Theorem of Algebra) If K is a real closed field then $K(\sqrt{-1})$ is algebraically closed.*

Proof. Let $L \supseteq K(\sqrt{-1})$ be an algebraic extension of $K(\sqrt{-1})$. We show $L = K(\sqrt{-1})$.

Set $G := \mathrm{Gal}(L/K)$. Then $[L : K] = |G| = 2^a m$, $a \geqslant 1$, m odd.

Let $S < G$ be a 2-Sylow subgroup ($|S| = 2^a$), and $F := \mathrm{Fix}(S)$. We have

$$[F : K] = [G : S] = m \qquad \text{odd}.$$

Since K is real closed, it follows that $m = 1$, so $G = S$ and $|G| = 2^a$. Now

$$[L : K(\sqrt{-1})][K(\sqrt{-1}) : K] = [L : K] = 2^a.$$

Therefore $[L : K(\sqrt{-1})] = 2^{a-1}$. We claim that $a = 1$.

If not, set $G_1 := \mathrm{Gal}(L/K(\sqrt{-1}))$, let S_1 be a subgroup of G_1 of index 2, and $F_1 := \mathrm{Fix}(S_1)$. So

$$[F_1 : K(\sqrt{-1})] = [G_1 : S_1] = 2,$$

and F_1 is a quadratic extension of $K(\sqrt{-1})$. But every element of $K(\sqrt{-1})$ is a square by Lemma 2.1, contradiction. □

Notation. We denote by $\bar{K}$ the algebraic closure of a field K, i.e. the smallest algebraically closed field containing K.

We have just proved that if K is real closed then $\bar{K} = K(\sqrt{-1})$.

3. Factorization in $R[\mathrm{x}]$

Corollary 3.1. *(Irreducible elements in $R[\mathrm{x}]$ and prime factorizaction in $R[\mathrm{x}]$). Let R be a real closed field, $f(\mathrm{x}) \in R[\mathrm{x}]$. Then*

(1) *if $f(\mathrm{x})$ is monic and irreducible then*
$$f(\mathrm{x}) = \mathrm{x} - a \quad or \quad f(\mathrm{x}) = (\mathrm{x} - a)^2 + b^2, \quad b \neq 0;$$
(2)
$$f(\mathrm{x}) = d \prod_{i=1}^{n} (\mathrm{x} - a_i) \prod_{j=1}^{m} (\mathrm{x} - d_j)^2 + b_j^2, \quad b_j \neq 0.$$

Proof. Let $f(\mathrm{x}) \in R[\mathrm{x}]$ be monic and irreducible. Then $\deg(f) \leqslant 2$.

Suppose not, and let $\alpha \in \bar{R}$ a root of $f(\mathrm{x})$. Then
$$[R(\alpha) : R] = \deg(f) > 2.$$

On the other hand, by 2.2
$$[R(\alpha) : R] \leqslant [\bar{R} : R] = 2,$$

contradiction.

If $\deg(f) = 1$, then $f(\mathrm{x}) = \mathrm{x} - a$, for some $a \in R$.

If $\deg(f) = 2$, then $f(\mathrm{x}) = \mathrm{x}^2 - 2a\mathrm{x} + c = (\mathrm{x} - a)^2 + (c - a^2)$, for some $a, c \in R$.

We claim that $c - a^2 > 0$. If not,
$$c - a^2 \leqslant 0 \;\Rightarrow\; -(c - a^2) \geqslant 0 \;\Rightarrow\; a^2 - c \geqslant 0,$$

the discriminant $4(a^2 - c) \geqslant 0$, $f(\mathrm{x})$ has a root in R and factors, contradiction.

Therefore $(c - a^2) \in R^2$ and there is $b \in R$ such that $(c - a^2) = b^2 \neq 0$. □

Corollary 3.2. *(Zwischenwertsatz : Intermediate value Theorem) Let R be a real closed field, $f(\mathrm{x}) \in R[\mathrm{x}]$. Assume $a < b \in R$ with $f(a)f(b) < 0$. Then $\exists\, c \in R$, $a < c < b$ such that $f(c) = 0$.*

Proof. We can assume $f(a) < 0 < f(b)$.

By previous Corollary,

$$\begin{aligned} f(\mathrm{x}) &= d\prod_{i=1}^{n}(\mathrm{x}-a_i)\prod_{j=1}^{m}(\mathrm{x}-d_j)^2+b_j^2 \\ &= d\prod_{i=1}^{n} l_i(\mathrm{x})q(\mathrm{x}), \end{aligned}$$

where $l_i(\mathrm{x}) := \mathrm{x} - a_i$, $\forall\, i = 1, \ldots, n$ and $q(\mathrm{x}) := \prod_{j=1}^{m}(\mathrm{x}-d_j)^2 + b_j^2$.

We claim that there is some $k \in \{1, \ldots, n\}$ such that $l_k(a)l_k(b) < 0$. Since

$$\operatorname{sign}(f) = \operatorname{sign}(d)\prod_{i=1}^{n}\operatorname{sign}(l_i)\operatorname{sign}(q) \quad \text{and} \quad \operatorname{sign}(q) = 1,$$

if we had that

$$\operatorname{sign}(l_i(a)) = \operatorname{sign}(l_i(b)) \quad \forall\, i \in \{1, \ldots, n\},$$

we would have

$$\operatorname{sign}(f(a)) = \operatorname{sign}(f(b)),$$

in contradiction with $f(a)f(b) < 0$.

For such a k,

$$l_k(a) < 0 < l_k(b),$$

i.e.

$$a - a_k < 0 < b - a_k,$$

and $c := a_k \in\,]a, b[$ is a root of $f(\mathrm{x})$. □

Corollary 3.3. *(Rolle) Let R be a real closed field, $f(\mathrm{x}) \in R[\mathrm{x}]$, Assume that $a, b \in R$, $a < b$ and $f(a) = f(b) = 0$. Then $\exists\, c \in R$, $a < c < b$ such that $f'(c) = 0$.*