REAL ALGEBRAIC GEOMETRY LECTURE NOTES (16: 10/12/09)

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Contents

Let R be a real closed field.

1. Cylindrical algebraic decomposition

Theorem 1.1. *Let* $\underline{\mathrm{x}} = (\mathrm{x}_1, \ldots, \mathrm{x}_n)$. *Let* $f_1(\underline{\mathrm{x}}, \mathrm{y}), \ldots, f_s(\underline{\mathrm{x}}, \mathrm{y})$ *be polynomials in* $n+1$ *variables with coefficients in* R. *Then there exixts a partition of* R^n *into a finite number of semialgebraic sets*

$$R^n = A_1 \dot\cup \cdots \dot\cup A_m$$

and for each $i = 1, \ldots, m$ *there exists a finite number (possibly* 0*) of continuous semialgebraic functions* $\xi_{i1}, \ldots, \xi_{il_i}$ *defined on* A_i *with*

$$\xi_{i1} < \cdots < \xi_{il_i}$$

$$\xi_{ij} \colon A_i \longrightarrow R$$

and $\xi_{ij}(\underline{x}) < \xi_{ij+1}(\underline{x})$ *for all* $\underline{x} \in A_i$, *for all* $j = 1, \ldots, l_i$, *such that*

(*i*) *for each* $\underline{x} \in A_i$, $\{\xi_{i1}(\underline{x}), \ldots, \xi_{il_i}(\underline{x})\} = \{$*roots of those polynomials among* $f_1(\underline{x}, \mathrm{y}), \ldots, f_s(\underline{x}, \mathrm{y})$ *which are not identically zero*$\}$;

(*ii*) *for each* $\underline{x} \in A_i$ *and* $y \in R$, $\operatorname{sign}(f_1(\underline{x}, y)), \ldots, \operatorname{sign}(f_s(\underline{x}, y))$ *depend only on* $\operatorname{sign}(y - \xi_{i1}), \ldots, \operatorname{sign}(y - \xi_{il_i})$.

We will prove this Theorem using the following Proposition:

Proposition 1.2. *(Main proposition "with coefficients")*
Let $f_1(\underline{\mathrm{x}}, \mathrm{y}), \ldots, f_s(\underline{\mathrm{x}}, \mathrm{y})$ *be polynomials in* $n+1$ *variables with coefficients in* R. *Let* $q := \max_{i=1,\ldots,s}\{\deg$ *in* y *of* $f_i(\underline{\mathrm{x}}, \mathrm{y})\}$ *and* $w \in W_{s,q}$.

Then there exists a boolean combination $B_w(\underline{\mathrm{x}})$ *of polynomial equations and inequalities in the variables* $\underline{\mathrm{x}}$ *with coefficients in* R *such that for any* $\underline{x} \in R^n$,

$$\operatorname{sign}_R(f_1(\underline{x}, \mathrm{y}), \ldots, f_s(\underline{x}, \mathrm{y})) = w \iff B_w(\underline{x}) \text{ is satisfied in } R.$$

Proof. Let $\underline{a} \in R^p$ be the list of coefficients of the polynomials $f_1, \dots, f_s$. Then for every $k = 1, \dots, s$,

$$f_k(\underline{\mathrm{x}}, \mathrm{y}) = F_k(\underline{a}, \underline{\mathrm{x}}, \mathrm{y}),$$

where $F_k(\underline{\mathrm{t}}, \underline{\mathrm{x}}, \mathrm{y}) \in \mathbb{Z}[\underline{\mathrm{t}}, \underline{\mathrm{x}}, \mathrm{y}]$ is a polynomial in $p + n + 1$ variables.

Then there is a boolean combination $B^*_w(\underline{\mathrm{t}}, \underline{\mathrm{x}})$ of polynomial equations and inequalities in the variables $(\underline{\mathrm{t}}, \underline{\mathrm{x}})$ with coefficients in $\mathbb{Z}$ such that, for every $(\underline{t}, \underline{x}) \in R^{p+n}$, we have

$$\mathrm{sign}_R(F_1(\underline{t}, \underline{x}, \mathrm{y}), \dots, F_s(\underline{t}, \underline{x}, \mathrm{y})) = w \;\Leftrightarrow\; B^*_w(\underline{t}, \underline{x}) \text{ holds.}$$

Now set $B_w(\mathrm{x}) = B^*_w(\underline{a}, \underline{\mathrm{x}})$. □

Let us prove now Theorem 1.1:

Proof of the Theorem. Without loss of generality we may assume that the set $\{f_1, \dots, f_s\}$ is closed under derivation with respect to the variable y (because we can always remove the functions ξ_{ij} that do not give the roots of the polynomials belonging to the initial family, and the conclusions of the theorem still hold with the remaining ξ_{ij}'s).

As in the previous Proposition, let $q := \max_{i=1,\dots,s}\{\deg \text{ in } y \text{ of } f_i(\underline{x}, y)\}$. Now $W_{s,q}$ is a finite set with

$$|W_{s,q}| = 3^{sq}.$$

For $w \in W_{s,q}$, define:

$$\begin{aligned} A_w : &= \{\underline{x} \in R^n : B_w(\underline{x}) \text{ is satisfied }\} \\ &= \{\underline{x} \in R^n : \mathrm{sign}_R(f_1(\underline{x}, \mathrm{y}), \dots, f_s(\underline{x}, \mathrm{y})) = w\}. \end{aligned}$$

Observe that A_w is a semialgebraic set of R^n. Let $A_1, \dots, A_m$ be the semialgebraic sets among the A_w that are non-empty, i.e.

$$\{A_1, \dots, A_m\} = \{A_w : w \in W_{s,q} \text{ and } A_w \neq \emptyset\}.$$

Note that by definition of A_w we have that $A_1, \dots, A_m$ form a partition of R^n (they are all disjoint because $w_1 \neq w_2 \Rightarrow A_{w_1} \cap A_{w_2} = \emptyset$, and for every $\underline{x} \in R^n$, $\underline{x} \in A_w$ with $w = \mathrm{sign}_R(f_1(\underline{x}, \mathrm{y}), \dots, f_s(\underline{x}, \mathrm{y})))$.

Note also that by definition of A_w, $\mathrm{sign}_R(f_1(\underline{x}, \mathrm{y}), \dots, f_s(\underline{x}, \mathrm{y})) = w \in W_{q,s}$ is constant on each A_i. In other words by definition of w there is a number $l_i \leqslant sq$ such that, for each $\underline{x} \in A_i$, the polynomials among $f_1(\underline{x}, \mathrm{y}), \dots, f_s(\underline{x}, \mathrm{y})$ which are not identically zero have altoghether l_i roots

$$\xi_{i1}(\underline{x}) \;<\; \cdots \;<\; \xi_{il_i}(\underline{x})$$

and moreover for every $k = 1, \dots, s$ the signs

$$\mathrm{sign}(f_k(\underline{x},\, \xi_{ij}(\underline{x}))), \qquad j = 1, \dots, l_i$$

$$\mathrm{sign}(f_k(\underline{x},\,]\, \xi_{ij}(\underline{x}), \xi_{ij+1}(\underline{x})\, [)), \qquad j = 0, \dots, l_i$$

depend only on i and not on $\underline{x} \in A_i$ (with the convention $\xi_{i0} = -\infty$ and $\xi_{il_{i+1}} = +\infty$).

Now it remains to show that each ξ_{ij} is semialgebraic and continuous.

The graph of ξ_{ij} is

$$\Gamma(\xi_{ij}) = \{(\underline{x}, y) \in A_i \times R \ : \exists\, (y_1, \ldots, y_{l_i}) \in R^{l_i} (\prod_k f_k(\underline{x}, y_1) = \cdots = \prod_k f_k(\underline{x}, y_{l_i}) = 0$$
$$\text{and } y_1 < \cdots < y_{l_i} \text{ and } y = y_j)\}$$

(where k ranges over the subscripts of those polynomials $f_k(\underline{x}, \mathrm{y})$ that are not identically zero on A_i), and therefore the function ξ_{ij} is semialgebraic.

To show the continuity of ξ_{ij}, fix $\underline{x}_0 \in A_i$. Then $y_j = \xi_{ij}(\underline{x}_0)$ is a simple root of at least one of $\{f_1(\underline{x}_0, \mathrm{y}), \ldots, f_s(\underline{x}_0, \mathrm{y})\}$ (closure under derivatives of the family), say of $f_1(\underline{x}_0, \mathrm{y})$. For $\varepsilon \in R$ small enough,

$$f_1(\underline{x}_0, y_j - \varepsilon) f_1(\underline{x}_0, y_j + \varepsilon) < 0.$$

Hence, in a neighbourhood U of $\underline{x}_0$ in R^n, we have

$$\forall\, \underline{x} \in U \qquad f_1(\underline{x}, y_j - \varepsilon) f_1(\underline{x}, y_j + \varepsilon) < 0$$

and $f_1(\underline{x}, \mathrm{y})$ has a root between $y_j - \varepsilon$ and $y_j + \varepsilon$ is $\xi_{ij}(\underline{x})$. This proves that ξ_{ij} is continuous. □