REAL ALGEBRAIC GEOMETRY LECTURE NOTES (20: 07/01/10)

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Contents

Let R be a real closed field.

1. Recall and plan

During the last lecture we proved that:

Proposition 1.1. *Let $\varphi\colon]0,r[\, \to R$ be a continuous bounded semialgebraic function defined on an interval $]0,r[\, \subset R$. Then φ can be continuously extended to* 0.

This was done assuming the following Lemma that we did not yet prove:

Lemma 1.2. *Let $A \subseteq R$ be a semialgebraic set, $\varphi\colon A \to R$ a semialgebraic function. Then there exists a nonzero polynomial $f \in R[\mathrm{x},\mathrm{y}]$ such that for every $x \in A$, $f(x,\varphi(x)) = 0$.*

We shall postpone the proof of the previous Lemma to next lecture, since we want to focus today on the proof of the Curve Selection Lemma. For this we shall further assume Thom's Lemma:

Proposition 1.3. *(Thom's Lemma) Let $f_1,\dots,f_s$ be a family of polynomials in $R[\mathrm{x}]$ closed under derivation. Let $\varepsilon\colon \{1,\dots,s\} \to \{-1,0,1\}$ be a sign condition. Set*

$$A_\varepsilon := \bigcap_{k=1}^{s} \{x \in R : \operatorname{sign}(f_k(x)) = \varepsilon(k)\}.$$

Denote by $A_{\bar\varepsilon}$ the semialgebraic subset of R obtained by relaxing the strict inequalities in A_ε, i.e. :

$$A_{\bar\varepsilon} := \bigcap_{k=1}^{s} \{x \in R : \operatorname{sign}(f_k(x)) = \overline{\varepsilon(k)}\}.$$

where $\bar\varepsilon$ is defined as follows:

$$\bar 0 = \{0\} \qquad \overline{-1} = \{-1,0\} \qquad \bar 1 = \{0,1\}$$

Then

(*i*) *either* A_ε *is empty, or* A_ε *is a point, or* A_ε *is an open interval;*
(*ii*) *if* A_ε *is nonempty then its closure is* $A_{\bar\varepsilon}$*;*
(*iii*) *if* A_ε *is empty then* $A_{\bar\varepsilon}$ *is either empty or a point.*

Using Prop 1.1 (proved last time) and Thom's Lemma (to be proved next time) our goal today is to prove the following:

Theorem 1.4. *(**Curve Selection Lemma**) Let A be a semialgebraic subset of R^n, $x \in R^n$, $x \in \bar{A} = \operatorname{clos}(A)$. Then there exists a continuous semialgebraic map $f\colon [\,0,1\,] \to R^n$ such that $f(0) = x$ and $f(]\,0,1\,]) \subset A$.*

Lemma 1.5. *Let $f_1, \dots, f_s \in R[\mathrm{x}_1, \dots, \mathrm{x}_n; \mathrm{y}]$ be **quasi-monic** with respect to* y *(i.e. $f_k = a_{d_k}\mathrm{y}^{d_k} + g_{d_k}(\mathrm{x}_1, \dots, \mathrm{x}_n)\mathrm{y}^{d_k-1} + \cdots + g_0(\mathrm{x}_1, \dots, \mathrm{x}_n)$ and $a_{d_k} \in R$ is constant). Assume that the set $\{f_1, \dots, f_s\}$ is closed under derivation with respect to* y*.*

Let $(A_i\ ;\ (\xi_{ij})_{j=1,\dots,l_i})_{i=1,\dots,m}$ be a slicing of $\{f_1, \dots, f_s\}$. Then every function ξ_{ij} can be continuously extended to the closure of A_i.

We shall prove the CSL and Lemma 1.5 simultaneously by induction on n in the following way. We shall show that:

(*i*) CSL is true for $n = 1$.
(*ii*) CSL for n implies Lemma 1.5 for n.
(*iii*) CSL and Lemma 1.5 for n imply CSL for $n + 1$.

(Clearly once (*i*), (*ii*), (*iii*) are established, CSL and Lemma 1.5 will follow by induction).

2. Proof of the Curve Selection Lemma

(*i*) $n = 1$. Let $x \in \bar{A}$. We may assume $x \notin A$ (otherwise take f to be the constant map $f\colon [0,1] \to R^n$, $f(r) = x\ \forall\, r$).

(By o-minimality) we know that $A \subset R$ semialgebraic is a finite union of intervals and points. So the result is clear in this case (if $x \in \bar{A}$, say x is the endpoint of a (half) open interval I of the form $(x, b] \subset A$ or $(x, b) \subset A$ or $[a, x) \subset A$ or (a, x), in all cases one can define continuous semialgebraic $f\colon [0,1] \to I$ with $f(0) = x$).

(*ii*) Assume CSL holds for n. We show that Lemma 1.5 holds for n.

For fixed i, j and $\underline{x} \in A_i$, we set

$$\varepsilon(k) := \operatorname{sign}(f_k(\underline{x}, \xi_{ij}(\underline{x}))),$$

with $k = 1, \dots, s$. This is well-defined since $\operatorname{sign}(f_k(x, \xi_{ij}(x)))$ does not depend on $\underline{x} \in A_i$.

Let $\underline{x}' \in \operatorname{clos}(A_i)$. We show that ξ_{ij} can be continuously extended to the semialgebraic set $A_i \cup \{\underline{x}'\}$.

By CSL for n there is $f\colon [0,1] \to R^n$ countinuous and semialgebraic such that $f(0) = \underline{x}'$ and $f(]0,1]) \subset (A_i \cap \bar{B}_n(\underline{x}', 1)) = A$, where $\bar{B}_n(\underline{x}', 1)$ is the n-dimensional closed ball with center $\underline{x}'$ and radius 1, i.e.

$$\bar{B}_n(\underline{x}', 1)) = \{\underline{a} \in R^n \mid \|\underline{a} - \underline{x}'\| \leqslant 1\},$$

which is a closed and bounded semialgebraic set.

Define $\varphi\colon]0,1] \to R$, $\varphi := (\xi_{ij} \circ f_{|]0,1]})$. Then φ is continuous and semialgebraic. We want to show that φ is bounded in order to apply Prop 1.1.

Now let $k \in \{1, \ldots, s\}$ be such that for $\underline{x} \in A_i$:

$$\xi_{ij}(\underline{x}) \text{ is a root of } f_k(\underline{x}, \mathrm{y}),$$

i.e. say for $\underline{x} \in A_i$, $\xi_{ij}(\underline{x})$ is a root of

$$f_k(\underline{x}, \mathrm{y}) = a_d \mathrm{y}^d + g_{d-1}(\underline{x})\mathrm{y}^{d-1} + \cdots + g_0(\underline{x})$$

By Corollary 2.1 of Lecture 6 we have for $\underline{x} \in A_i$:

$$|\xi_{ij}(\underline{x})| \leqslant 1 + |\frac{g_{d-1}(\underline{x})}{a_d}| + \cdots + |\frac{g_0(\underline{x})}{a_d}|$$

Consider now $\underline{x}$ in the bounded set $A_i \cap \bar{B}_n(\underline{x}', 1))$.

Each polynomial $g_0, \ldots, g_{d-1}$ is bounded on this set.

So let $a \in R$ be such that for every $\underline{x} \in A_i \cap \bar{B}_n(\underline{x}', 1))$ we have

$$|g_l(\underline{x})| \leqslant a \qquad \forall\, l = 0, \ldots, d-1.$$

Therefore φ is a bounded function. Indeed let $t \in]0,1[$ and compute

$$|\varphi(t)| = |\xi_{ij}(f(t))| \qquad \text{with } \underline{x} = f(t) \in A_i \cap \bar{B}_n(\underline{x}', 1))$$

so

$$|\xi_{ij}(f(t))| \leqslant 1 + |g_{d-1}(f(t))| + \cdots + |g_0(f(t))| \leqslant 1 + \frac{a}{|a_d|} + \cdots + \frac{a}{|a_d|} = 1 + \frac{da}{|a_d|}.$$

We apply Proposition 1.1 to the bounded continuous semialgebraic function φ to extend φ continuously to 0 and we define now

$$\xi_{ij}(\underline{x}') := \varphi(0).$$

Claim. ξ_{ij} is continuous at x'.
We argue by contradiction. If not $\exists\, \mu > 0$, $\mu \in R$ such that

$$\forall \eta \in R\ \exists\, \underline{x} \in A_i \text{ such that } \|\underline{x} - \underline{x}'\| < \eta \text{ but } |\xi_{ij}(\underline{x}) - \varphi(0)| \geqslant \mu.$$

Consider

$$C_\mu = \{\underline{x} \in A_i \mid |\xi_{ij}(\underline{x}) - \varphi(0)| \geqslant \mu\} \cap \bar{B}_n(\underline{x}', 1)$$

Since $\underline{x}' \in \operatorname{clos}(C_\mu) \subset R^n$, we can apply CSL to have a continuous semialgebraic function

$$g\colon [0,1] \longrightarrow R^n$$

with $g(0) = \underline{x}'$ and $g(]0,1]) \subset C_\mu$. We now consider

$$\psi\colon]0,1] \to R, \quad \psi := (\xi_{ij} \circ g_{|]0,1]}).$$

As before ψ can be continuously extended to 0.

Subclaim.

($\bullet$) $|\varphi(0) - \psi(0)| \geqslant \mu$.

($\bullet\bullet$) For every $k = 1, \dots, s$

$$\operatorname{sign} f_k(\underline{x}', \varphi(0)) \in \overline{\varepsilon(k)}$$

$$\operatorname{sign} f_k(\underline{x}', \psi(0)) \in \overline{\varepsilon(k)}.$$

Proof of the Subclaim.

($\bullet$) For every $t \in]0,1]$, $\psi(t) = \xi_{ij}(g(t)) = \xi_{ij}(\underline{x})$ for some $\underline{x} \in C_\mu$. Therefore $|\varphi(t) - \psi(0)| \geqslant \mu$ for every $t \in]0,1]$ and by continuity of ψ, $|\varphi(0) - \psi(0)| \geqslant \mu$.

($\bullet\bullet$) Let $k \in \{1, \dots, s\}$.

If $\varepsilon(k) = 0$, then $f_k(\underline{x}, \xi_{ij}(\underline{x})) = 0$ for all $\underline{x} \in A_i$, so by continuity

$$\begin{cases} f_k(\underline{x}',\, \varphi(0)) = 0 \qquad \text{and} \\ \\ f_k(\underline{x}',\, \psi(0)) = 0. \end{cases}$$

Similarly if $\varepsilon(k) = -1$, then $f_k(\underline{x}, \xi_{ij}(\underline{x})) < 0$ for all $\underline{x} \in A_i$, so by continuity

$$\begin{cases} f_k(\underline{x}',\, \varphi(0)) \geqslant 0 \qquad \text{and} \\ \\ f_k(\underline{x}',\, \psi(0)) \geqslant 0. \end{cases}$$

and finally if $\varepsilon(k) = 1$, then $f_k(\underline{x}, \xi_{ij}(\underline{x})) > 0$ for all $\underline{x} \in A_i$ and

$$\begin{cases} f_k(\underline{x}',\, \varphi(0)) \geqslant 0 \qquad \text{and} \\ \\ f_k(\underline{x}',\, \psi(0)) \geqslant 0. \end{cases}$$

□

Consider now the set

$$\{y \in R \,|\, \operatorname{sign}(f_k(\underline{x}', y)) \in \bar{\varepsilon}(k),\, k = 1, \dots, s\}.$$

By Thom's Lemma this set is either empty or reduces to a point. On the other hand $\varphi(0) \neq \psi(0)$ and bot $\varphi(0)$, $\psi(0)$ belong to this set by the subclaim, contradiction. Therefore ξ_{ij} is continuous at $\underline{x}'$.

(*iii*) We assume CSL and Lemma 1.5 to be true for n and show that CSL is true for $n + 1$.

Let $A \subseteq R^{n+1}$ semialgebraic given by a boolean combination of sign conditions on $f_1, \ldots, f_s \in R[\mathrm{x}_1, \ldots, \mathrm{x}_n, \mathrm{y}]$.

Claim. We may assume that $f_1, \ldots, f_s$ are quasi-monic and that the family is closed under derivation, so that $f_1, \ldots, f_s$ satisfy the conditions of Lemma 1.5.

Let $(A_i\ ; \{\xi_{ij}\}_{j=1,\ldots,l_i})_{i=1,\ldots m}$ be a slicing of $f_1, \ldots, f_s$. So $A_i \subset R^n$ for every $i = 1, \ldots, m$ and the set A is the union of the graphs of some functions ξ_{ij} and some slices $]\xi_{ij},\, \xi_{ij+1}[$.

Let $(\underline{x}, y) \in \mathrm{clos}(A) \subseteq R^{n+1}$. We have to consider the following cases:

(1) $(\underline{x}, y) \in \mathrm{clos}(\Gamma(\xi_{ij}))$, $\xi_{ij} \colon A_i \to R$.

(2) $(\underline{x}, y) \in \mathrm{clos}(\,]\xi_{ij},\, \xi_{ij+1}[\,)$, where $1 < j < l_i$.

(3) $(\underline{x}, y) \in \mathrm{clos}(\,]\xi_{ij},\, \xi_{ij+1}[\,)$, where $j = 1$ or $j = l_i$.

Case 1. Let $(\underline{x}, y) \in \mathrm{clos}(\Gamma(\xi_{ij}))$, $\xi_{ij} \colon A_i \to R$, with $\Gamma(\xi_{ij}) \subseteq A$. Applying the CSL, let $\varphi\colon [0,1] \to R^n$ be a continuous and semialgebraic map such that $\varphi(0) = \underline{x}$ and $\varphi(]0,1]) \subseteq A_i$.

We can use Lemma 1.5 for n to extend ξ_{ij} at $\underline{x}$ continuously. So we must have $\xi_{ij}(\underline{x}) = y$.

Now set

$$\psi\colon [0,\, 1] \xrightarrow{\ \varphi\ } A_i \cup \{\underline{x}\} \xrightarrow{\ \xi_{ij}\ } R$$

and $f := (\varphi, \psi)$. f is continuous semialgebraic, $f(0) = (\varphi(0), \psi(0)) = (\underline{x}, y)$ and $f(]0,\, 1]) \subseteq A$.

Case 2. $(\underline{x}, y) \in \mathrm{clos}(\,]\xi_{ij},\, \xi_{ij+1}[\,)$, where $1 < j < l_i$, with $]\xi_{ij},\, \xi_{ij+1}[\, \subseteq A \subseteq R^{n+1}$, $\xi_{ij}, \xi_{ij+1} \colon A_i \to R$.

By CSL for n let $\varphi\colon [0,\, 1] \to R^n$ be a continuous semialgebraic map with $\varphi(0) = \underline{x}$ and $\varphi(]0,1]) \subseteq A_i$.

By Lemma 1.5 for n extend the function ξ_{ij} and ξ_{ij+1} continuously to $\underline{x}$:

$$\begin{array}{rcll} \xi_{ij}\colon A_i \cup \{\underline{x}\} & \longrightarrow & R & \quad \xi_{ij}(\underline{x}) \in R \\ \xi_{ij+1}\colon A_i \cup \{\underline{x}\} & \longrightarrow & R & \quad \xi_{ij+1}(\underline{x}) \in R \end{array}$$

Set

$$t := \begin{cases} 1/2 & \text{if } \xi_{ij}(\underline{x}) = \xi_{ij+1}(\underline{x}) \\[2ex] \frac{y - \xi_{ij}(\underline{x})}{\xi_{ij+1}(\underline{x}) - \xi_{ij}(\underline{x})} & \text{if } \xi_{ij}(\underline{x}) \neq \xi_{ij+1}(\underline{x}) \end{cases}$$

and $\psi\colon [(1-t)\xi_{ij} + t(\xi_{ij+1})] \circ \varphi$. Then ψ is continuous semialgebraic and $\psi(0) = y$. Set $f := (\varphi, \psi)$. f is continuous and semialgebraic, with $f(0) = (\varphi(0), \psi(0)) = (\underline{x}, y)$ and $f(]0,\, 1]) \subseteq A$.

Case 3. Exercise.