REAL ALGEBRAIC GEOMETRY LECTURE NOTES (23: 19/01/10)

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Contents

PART III
Konvexe Bewertungen und reelle Stellen auf angeordnete Körper

1. Valued Z-modules and valued Q-vector spaces

All modules M considered are left Z-modules for a fixed ring Z with 1 (we are mainly interested in $Z = \mathbb{Z}$, i.e. in valued abelian groups).

Definition 1.1. Let Γ be a totally ordered set and ∞ an element greater than aech element of Γ (Notation: $\infty > \Gamma$). A surjective map

$$v \colon M \longrightarrow \Gamma \cup \{\infty\}$$

is a **valuation** on M (and (M, v) is a **valued module**) if $\forall\, x, y \in M$ and $\forall\, r \in Z$:

(i) $v(x) = \infty \;\Leftrightarrow\; x = 0$;

(ii) $v(rx) = v(x)$, if $r \neq 0$ (value preserving scalar multiplication);

(iii) $v(x - y) \geqslant \min\{v(x), v(y)\}$ (ultrametric Δ-inequality).

Remark 1.2. $(i) + (ii) \;\Rightarrow\; M$ is torsion-free.

Remark 1.3. Consequences of the ultrametric:

- $v(x) \neq v(y) \;\Rightarrow\; v(x + y) = \min\{v(x), v(y)\}$;
- $v(x + y) > v(x) \;\Rightarrow\; v(x) = v(y)$.

Definition 1.4. $v(M) := \Gamma = \{v(x) : 0 \neq x \in M\}$ is the **value set** of M.

Definition 1.5.

(*i*) Let (M_1, v_1), (M_2, v_2) two valued modules with value sets Γ_1 and Γ_2 respectively. Let

$$h \colon M_1 \longrightarrow M_2$$

be an isomorphism of Z-modules. We say that h **preserves the valuation** if there is an isomorphism of ordered sets

$$\varphi \colon \Gamma_1 \longrightarrow \Gamma_2$$

such that $\forall\, x \in M_1 : \ \varphi(v_1(x)) = v_2(h(x))$.

(*ii*) Two valuations v_1, v_2 on M are **equivalent** if the identity map on M preserves the valuation.

Definition 1.6.

(1) An **ordered system of Z-modules** is denoted by:

$$[\,\Gamma,\ \{B(\gamma) : \gamma \in \Gamma\}\,]$$

where $\{B(\gamma) : \gamma \in \Gamma\}$ is a family of modules indexed by a totally ordered set Γ.

(2) Two systems

$$S_i = [\,\Gamma_i,\ \{B_i(\gamma) : \gamma \in \Gamma_i\}\,] \qquad i = 1, 2$$

are **isomorphic** (we write $S_1 \cong S_2$) if and only if there are an isomorphism

$$\varphi \colon \Gamma_1 \longrightarrow \Gamma_2$$

of totally ordered sets, and $\forall\, \gamma \in \Gamma_1$ an isomorphism of modules

$$\varphi_\gamma \colon B_1(\gamma) \longrightarrow B_2(\varphi(\gamma)).$$

(3) Let (M, v) be a valued module, $\Gamma := v(M)$. For $\gamma \in \Gamma$ set

$$M^\gamma := \{x \in M : v(x) \geqslant \gamma\}$$

$$M_\gamma := \{x \in M : v(x) > \gamma\}.$$

Then $M_\gamma \subsetneq M^\gamma \subsetneq M$. Set

$$B(M, \gamma) := M^\gamma / M_\gamma.$$

$B(M, \gamma)$ is **the (homogeneous) component corresponding to** γ. **The skeleton** (*das skelett*) of the valued module (M, v) is the ordered system

$$S(M) := [\,v(M),\ \{B(M, \gamma) : \gamma \in v(M)\}\,].$$

We write $B(\gamma)$ for $B(M, \gamma)$ if the context is clear.

(4) For every $\gamma \in \Gamma$, the **coefficient map** (*Koeffizient Abbildung*)

$$\begin{array}{rcl} \pi^M(\gamma, -)\colon M^\gamma & \longrightarrow & B(\gamma) \\ x & \mapsto & x + M_\gamma \end{array}$$

is the canonical projection.
We write $\pi(\gamma, -)$ instead of $\pi^M(\gamma, -)$ if the context is clear.

Lemma 1.7. *The skeleton is an isomorphism invariant, i.e.*

$$\begin{array}{rcl} \textit{if} \quad (M_1, v_1) & \cong & (M_2, v_2), \\ \textit{then} \quad S(M_1) & \cong & S(M_2). \end{array}$$

Proof. Let $h\colon M_1 \to M_2$ be an isomorphism which preserves the valuation. Then

$$\tilde{h}\colon v(M_1) \longrightarrow v(M_2)$$

defined by

$$\tilde{h}(v_1(x)) := v_2(h(x))$$

is a well defined map and an isomorphism of totally ordered sets.
For $\gamma \in v(M_1)$ the map

$$h_\gamma\colon B_1(\gamma) \longrightarrow B_2(\tilde{h}(\gamma))$$

defined by

$$\pi^{M_1}(\gamma, x) \mapsto \pi^{M_2}(\tilde{h}(\gamma), h(x))$$

is well defined and an isomorphism of modules. □

2. Hahn valued modules

A system $[\Gamma,\ \{B(\gamma) : \gamma \in \Gamma\}]$ of torsion-free modules can be realized as the skeleton of a valued module through the following canonical construction:

Consider $\prod_{\gamma\in\Gamma} B(\gamma)$ the product module. For $s \in \prod_{\gamma\in\Gamma} B(\gamma)$ define

$$\text{support}(s) = \{\gamma \in \Gamma : s(\gamma) \neq 0\}.$$

The **Hahn sum** $\bigsqcup_{\gamma\in\Gamma} B(\gamma)$ is the submodule of $\prod_{\gamma\in\Gamma} B(\gamma)$ consisting of elements with finite support (i.e. $\bigoplus_{\gamma\in\Gamma} B$) endowed with the valuation:

$$v_{\min}\colon \bigsqcup_{\gamma\in\Gamma} B(\gamma) \longrightarrow \Gamma \cup \{\infty\}$$
$$v_{\min}(s) = \min \text{support}(s).$$

(convention: $\min \emptyset = \infty$).

The **Hann product** $\mathrm{H}_{\gamma\in\Gamma} B(\gamma)$ is the submodule of $\prod_{\gamma\in\Gamma} B(\gamma)$ consisting of the elements with well-ordered support equipped with $v_{\min}$.

We recall that a totally ordered set Γ is **well-ordered** if every non-empty subset of Γ has a least, or equivalently if every descending sequence of elements from Γ is finite.

3. Hahn Sandwich Proposition

Lemma 3.1.

(*i*) $\coprod_{\gamma\in\Gamma} B(\gamma)) \subseteq \mathrm{H}_{\gamma\in\Gamma} B(\gamma)$.

(*ii*)

$$\begin{aligned} S(\coprod_{\gamma\in\Gamma} B(\gamma)) &\cong [\,\Gamma, \{B(\gamma) : \gamma \in \Gamma\}\,] \\ &\cong S(\mathrm{H}_{\gamma\in\Gamma} B(\gamma)). \end{aligned}$$

We shall show that if $Z = Q$ is a field and (V, v) is a valued Q-vector space with skeleton $S(V) = [\,\Gamma, B(\gamma)\,]$, then

$$(\coprod B(\gamma), v_{\min}) \hookrightarrow (V, v) \hookrightarrow (\mathrm{H}_{\gamma\in\Gamma} B(\gamma), v_{\min}).$$