REAL ALGEBRAIC GEOMETRY LECTURE NOTES (24: 21/01/10)

SALMA KUHLMANN

Contents

1. Hahn Sandwich Proposition

From now, let $Z = Q$ be a field and (V, v) a valued Q-vector space with skeleton $S(V) = [\,\Gamma, B(\gamma)\,]$. We want to show

$$(\coprod_{\gamma\in\Gamma} B(\gamma),\ v_{\min}) \hookrightarrow (V, v) \hookrightarrow (\mathrm{H}_{\gamma\in\Gamma}\, B(\gamma),\ v_{\min}).$$

2. Immediate extensions

Definition 2.1. Let (V_i, v_i) be valued Q-vector spaces ($i = 1, 2$).

(1) Let $V_1 \subseteq V_2$ Q-subspace with $v_1(V_1) \subseteq v_2(V_2)$. We say that (V_2, v_2) is an **extension** of (V_1, v_1), and we write

$$(V_1, v_1) \ \subseteq\ (V_2, v_2),$$

if $v_{2|_{V_1}} = v_1$.

(2) If $(V_1, v_1) \subseteq (V_2, v_2)$, for $\gamma \in v_1(V_1)$ the map

$$\begin{array}{rcl} B_1(\gamma) & \longrightarrow & B_2(\gamma) \\ x + (V_1)_\gamma & \mapsto & x + (V_2)_\gamma \end{array}$$

is a natural identification of $B_1(\gamma)$ as a Q-subspace of $B_2(\gamma)$. The extension $(V_1, v_1) \subseteq (V_2, v_2)$ is **immediate** if $\Gamma := v_1(V_1) = v_2(V_2)$ and $\forall\, \gamma \in v_1(V_1)$

$$B_1(\gamma) = B_2(\gamma).$$

Equivalently, $(V_1, v_1) \subseteq (V_2, v_2)$ is immediate if $S(V_1, v_1) = S(V_2, v_2)$.

Lemma 2.2. *(Characterization of immediate extensions)*
The extension $(V_1, v_1) \subseteq (V_2, v_2)$ *is immediate if and only if*

$$\forall\, x \in V_2,\ x \neq 0,\ \exists\, y \in V_1 \ \text{ such that } \ v_2(x-y) > v_2(x).$$

Proof. We show that in a valued Q-vector space (V, v), for every $x, y \in V$

$$v(x-y) > v(x) \iff \begin{cases} (i) & \gamma = v(x) = v(y) \ \text{ and} \\ (ii) & \pi(\gamma, x) = \pi(\gamma, y). \end{cases}$$

($\Leftarrow$) Assume (i) and (ii). So $x, y \in V^\gamma$ and $x - y \in V_\gamma$.
Then $v(x-y) > v(x) = \gamma$.

($\Rightarrow$) Assume $v(x-y) > v(x)$. We show (i) and (ii).
If $v(x) \neq v(y)$, then $v(x-y) = \min\{v(x), v(y)\}$. In both cases $\min\{v(x), v(y)\} = v(x)$ and $\min\{v(x), v(y)\} = v(y)$ we have a contradiction. (ii) is analogue.

□

Example 2.3. $(\coprod_{\gamma\in\Gamma} B(\gamma),\ v_{\min}) \ \subseteq \ (\mathrm{H}_{\gamma\in\Gamma} B(\gamma),\ v_{\min})$
is an immediate extension.

Proof. Given $x \in \mathrm{H}_{\gamma\in\Gamma} B(\gamma)$, $x \neq 0$, set

$$\gamma_0 := \min \operatorname{support}(x) \qquad \text{and} \qquad x(\gamma_0) := b_0 \in B(\gamma).$$

Let $y \in \coprod_{\gamma\in\Gamma} B(\gamma)$ such that

$$y(\gamma) = \begin{cases} 0 & \text{if } \gamma \neq \gamma_0 \\ b_0 & \text{if } \gamma = \gamma_0. \end{cases}$$

Namely $y = b_0 \chi_{\gamma_0}$, where

$$\chi_{\gamma_0} : \Gamma \longrightarrow Q$$

$$\chi_{\gamma_0}(\gamma) = \begin{cases} 1 & \text{if } \gamma = \gamma_0 \\ 0 & \text{if } \gamma \neq \gamma_0. \end{cases}$$

Then $v_{\min}(x-y) > \gamma_0 = v_{\min}(x)$ (because $(x-y)(\gamma_0) = x(\gamma_0) - y(\gamma_0) = b_0 - b_0 = 0$).

□

3. Valuation independence

Definition 3.1. $\mathcal{B} = \{x_i : i \in I\} \subseteq V \setminus \{0\}$ is **Q-valuation independent** if for $q_i \in Q$ with $q_i = 0$ for all but finitely many $i \in I$, we have

$$v(\sum_{i\in I} q_i x_i) = \min_{i \in I, q_i \neq 0} \{v(x_i)\}.$$

Remark 3.2. $\mathcal{B} \subseteq V \setminus \{0\}$ Q-valuation independent $\Rightarrow$ Q-linear independent.
(Otherwise $\exists\ q_i \neq 0$ with $\sum q_i x_i = 0$ and $v(\sum q_i x_i) = \infty$).

Proposition 3.3. *(Characterization of valuation independence)*
Let $\mathcal{B} \subseteq V \setminus \{0\}$. Then $\mathcal{B}$ is Q-valuation independent if and only if $\forall\, n \in \mathbb{N}$, $\forall\, b_1, \ldots, b_n \in \mathcal{B}$ pairwise distinct with $v(b_1) = \cdots = v(b_n) = \gamma$, the coefficients

$$\pi(\gamma, b_1), \ldots, \pi(\gamma, b_n) \in B(\gamma)$$

are Q-linear independent in $B(\gamma)$.

Proof.

($\Rightarrow$) Let $b_1, \ldots, b_n \in \mathcal{B}$ with $v(b_1) = \cdots = v(b_n) = \gamma$ and suppose for a contradiction that

$$\pi(\gamma, b_1), \ldots, \pi(\gamma, b_n) \in B(\gamma)$$

are not Q-linear independent. So there are $q_1, \ldots, q_n \in Q$ non-zero such that $\pi(\gamma, \sum q_i b_i) = 0$ and $v(q_i b_i) > \gamma$, contradiction.

($\Leftarrow$) We show that

$$v(\sum q_i b_i) = \min\{v(b_i)\} = \gamma.$$

Since $\pi(\gamma, b_1), \ldots, \pi(\gamma, b_n)$ are Q-linear independent in $B(\gamma)$, also

$$\pi(\gamma, \sum_{i=0}^{n} q_i b_i) \neq 0,$$

i.e. $v(\sum q_i b_i) \leqslant \gamma$.
On the other hand $v(\sum q_i b_i) \geqslant \gamma$, so $v(\sum q_i b_i) = \gamma = \min\{v(b_i)\}$.

□

4. Maximal valuation independence

By Zorn's lemma, maximal valuation independent sets exist:

Corollary 4.1. *(Characterization of maximal valuation independent sets)*
$\mathcal{B} \subseteq V \setminus \{0\}$ is maximal valuation independent if and only if $\forall\, \gamma \in v(V)$

$$\mathcal{B}_\gamma := \{\pi(\gamma, b) : b \in \mathcal{B}; v(b) = \gamma\}$$

is a Q-vector space basis of $B(V, \gamma)$.

Corollary 4.2. *Let $\mathcal{B} \subseteq V \setminus \{0\}$ be valuation independent in (V, v). Then $\mathcal{B}$ is maximal valuation independent if and only if the extension*

$$\langle\, \mathcal{B} \,\rangle := (V_0, v_{|V_0}) \;\subseteq\; (V, v)$$

is an immediate extension.

Proof.

($\Rightarrow$) Assume $\mathcal{B} \subseteq V$ is maximal valuation independent. We show $V_0 \subseteq V$ is immediate.

If not $\exists\, x \in V$, $x \neq 0$ such that

$$\forall\, y \in V_0 : \; v(x-y) \leqslant v(x).$$

We will show that in this case $\mathcal{B} \cup \{x\}$ is valuation independent (which will contradict our maximality assumption).

Consider $v(y_0 + qx)$, $q \in Q$, $q \neq 0$, $y_0 \in V_0$. Set

$$y := -y_0/q.$$

We claim that $v(y_0 + qx) = v(x-y) = \min\{v(x), v(y)\}$

Fact.

$$v(x-y) \leqslant v(x) \iff v(x-y) = \min\{v(x), v(y)\}.$$

Proof of the fact. ($\Leftarrow$) is clear. To see ($\Rightarrow$), assume that $v(x-y) > \min\{v(x), v(y)\}$. If $\min\{v(x), v(y)\} = v(x)$, then we have a contradiction. If $\min\{v(x), v(y)\} = v(y) < v(x)$, then $v(x-y) = v(y) > v(y)$, again a contradiction.

($\Leftarrow$) Now assume $(V_0, v) \subseteq (V, v)$ is immediate. We show that $\mathcal{B}$ is maximal valuation independent.

If not, there is $\gamma \in v(V)$ such that B_γ is not a basis for $B(V, \gamma)$.

Let $b \in B(V, \gamma)$, $b \notin \langle \mathcal{B}_\gamma \rangle$.

$$b \in V^\gamma / V_\gamma \implies b = x + V_\gamma,$$

with $x \in V$, $v(x) = \gamma$.

Claim: $\forall\, y \in V_0$ $v(x-y) \leqslant v(x)$ (contradicting that the extension is immediate). This follows by Characterization of immediate extensions (Lemma 2.2).

$\square$

5. Valuation basis

Definition 5.1. $\mathcal{B}$ is a **Q-valuation basis** of (V, v) if

1. (1) $\mathcal{B}$ is a Q-basis,
2. (2) $\mathcal{B}$ is Q-valuation independent.

Remark 5.2. $\mathcal{B}$ Q-valuation basis $\Rightarrow$ $\mathcal{B}$ is maximal valuation independent.

Example 5.3. $(\coprod_{\gamma \in \Gamma} B(\gamma),\; v_{\min})$ admits a valuation basis.

Proof. Let $\mathcal{B}_\gamma$ be a Q-basis of $B(\gamma)$ for $\gamma \in \Gamma$ and consider

$$\mathcal{B} := \bigcup_{\gamma \in \Gamma} \{b\chi_{\{\gamma\}};\; b \in \mathcal{B}_\gamma\},$$

where $\forall\, \gamma \in \Gamma$

$$\chi_\gamma : \Gamma \longrightarrow Q$$

$$\chi_\gamma(\gamma') = \begin{cases} 1 & \text{if } \gamma = \gamma' \\ 0 & \text{if } \gamma \neq \gamma'. \end{cases}$$

□

Corollary 5.4. *(V, v) with skeleton $S(V) = [\,\Gamma, B(\gamma)\,]$ admits a valuation basis if and only if*

$$(V, v) \;\cong\; (\bigsqcup_{\gamma\in\Gamma} B(\gamma),\; v_{\min}).$$

Proof.

($\Leftarrow$) Clear.

($\Rightarrow$) Let $\mathcal{B}$ be a valuation basis for (V, v). Then $\mathcal{B} = \{b_i : i \in I\}$ is maximal valuation independent. For every $b_i \in \mathcal{B}, v(b_i) = \gamma$, define

$$h(b_i) = \pi(\gamma, b_i)\chi_\gamma$$

and extend it to V by linearity (note that $v(b_i) = v_{\min}(h(b_i))$).

□

Corollary 5.5. *Assume $S(V) = [\,\Gamma, B(\gamma)\,]$. Then*

$$(\bigsqcup_{\gamma\in\Gamma} B(\gamma),\; v_{\min}) \;\hookrightarrow\; (V, v).$$

Proof. By Zorn's lemma, let $\mathcal{B} \subset V \setminus \{0\}$ be maximal valuation independent. Set

$$V_0 := {}_Q\langle\,\mathcal{B}\,\rangle.$$

Then $\mathcal{B}$ is a valuation basis for V_0 and $V_0 \subseteq V$ (immediate), so $S(V_0) = S(V) = [\,\Gamma, B(\gamma)\,]$ and

$$(V_0, v) \;\cong\; (\bigsqcup_{\gamma\in\Gamma} B(\gamma),\; v_{\min}).$$

□