REAL ALGEBRAIC GEOMETRY LECTURE NOTES (26: 28/01/10)

SALMA KUHLMANN

Contents

1. Pseudo-completeness

Let (V, v) be a valued Q-vector space. We recall that

- (V, v) is said to be **maximally valued** if it admits no proper immediate extension.
- (V, v) is **pseudo-complete** if every pseudo-convergent sequence in V has a pseudo-limit in V.

Theorem 1.1. *(V, v) is maximally valued if and only if (V, v) is pseudo-complete.*

We prove only one implication:

(V, v) pseudo-complete $\Rightarrow$ (V, v) maximally valued.

This implication follows from the following:

Proposition 1.2. *Let (V, v) be an immediate extension of (V_0, v). Then any element in V which is not in V_0 is a pseudo-limit of a pseudo-Cauchy sequence of elements of V_0, without a pseudo-limit in V_0.*

Proof. Let $z \in V \setminus V_0$. Consider the set

$$X = \{v(z - a) : a \in V_0\}.$$

Since $z \notin V_0$, $\infty \notin X$.

We show that X cannot have a maximal element. Otherwise, assume $a_0 \in V_0$ and $v(z - a_0)$ maximal in X. Since the extension is immediate, by Lemma 2.2 of Lecture 24 there is $a_1 \in V_0$ such that $v(z-a_0-a_1) > v(z-a_0)$. So $a_0 + a_1 \in V_0$ and $v(z-(a_0+a_1)) > v(z-a_0)$, contradiction. Then X has no greatest element.

Select from X a well ordered cofinal subset $\{\alpha_\rho\}_{\rho\in\lambda}$. Since the set X has no greatest member, also $\{\alpha_\rho\}_{\rho\in\lambda}$ does not have a last term (see Lemma 4.3 of Lecture 25).

For every $\rho \in \lambda$ choose an element $a_\rho \in V_0$ with
$$v(z - a_\rho) = \alpha_\rho.$$

The identity
$$a_\sigma - a_\rho = (z - a_\rho) - (z - a_\sigma)$$

together with the inequality
$$v(z - a_\rho) < v(z - a_\sigma) \qquad (\forall\, \rho < \sigma \in \lambda)$$

imply
$$(*) \qquad v(a_\sigma - a_\rho) = v(z - a_\rho).$$

Then $\{a_\rho\}_{\rho\in\lambda}$ is pseudo-convergent with z as a pseudo-limit.
Suppose now that $\{a_\rho\}_{\rho\in\lambda}$ had a further limit $z_1 \in V_0$.
Then by Lemma 3.6 of Lecture 25 we have
$$v(z - z_1) > v(a_\sigma - a_\rho).$$

Combining this with $(*)$ we get
$$v(z - z_1) > v(z - a_\rho) = \alpha_\rho \qquad \forall\, \rho \in \lambda$$

and this is a contradiction, since $\{\alpha_\rho\}_{\rho\in\lambda}$ is cofinal in X. □

Theorem 1.3. *Suppose that*

(*i*) *V_i and V_i' are Q-valued vector spaces and V_i' is an immediate extension of V_i, for $i = 1, 2$.*

(*ii*) *h is an isomorphism of valued vector spaces of V_1 onto V_2.*

(*iii*) *V_2' is pseudo-complete.*

Then there exists an embedding h' of valued vector spaces of V_1' in V_2' such that h' extends h.
Moreover h' is an isomorphism of valued vector spaces of V_1' onto V_2' if and only if V_1' is pseudo-complete.

Proof. The picture is the following:

$$\begin{array}{ccc} V_1' & \overset{h'}{\hookrightarrow} & V_2' \\ \text{immediate}\Big| & & \Big|\text{immediate} \\ V_1 & \underset{\sim}{\overset{h}{\longrightarrow}} & V_2 \end{array}$$

By Zorn's Lemma, let
$$V_1 \subseteq M_1 \subseteq V_1',$$
$$V_2 \subseteq M_2 \subseteq V_2'$$

and g a valuation isomorphism of M_1 onto M_2 extending h. We shall show how to extend g to V_1'.

Let $y_1 \in V_1' \setminus M_1$. Since V_1' is an immediate extension of M_1 there exists a pseudo-convergent sequence

$$S = \{a_\rho\}_{\rho \in \lambda}$$

of M_1 without a pseudo-limit in M_1 but with a pseudo-limit $y_1 \in V_1'$. Consider

$$g(S) = \{g(a_\rho)\}_{\rho \in \lambda}$$

Since g is a valuation preserving isomorphism, $g(S)$ is a pseudo-convergent sequence of M_2 without a pseudo-limit in M_2 but with pseudo-limit $y_2 \in V_2'$, because V_2' is pseudo-complete.

Let $M_i' = \langle M_i, y_i \rangle$, for $i = 1, 2$, and denote by g' the unique Q-vector space isomorphism of the linear space M_1' onto the linear space M_2' extending g and such that $g'(y_1) = y_2$.

We show that g' is valuation preserving: let

$$y = x + qy_1 \qquad x \in M_1 \qquad q \in Q \setminus \{0\}$$

be an arbitrary element of $M_1' \setminus V_1$. The set

$$S(y) = \{x + qa_\rho\}_{\rho \in \lambda}$$

is a pseudo-convergent sequence in M_1 with pseudo-limit $y \in M_1$ and 0 is not a pseudo-limit (otherwise $-x/q \in M_1$ would be a pseudo-limit of S).

It follows that (since $y = x + qy_1$ is a pseudo-limit for the sequence $x + qa_\rho$ which does not have 0 as a pseudo-limit)

$$v(y) = \operatorname{Ult} S(y)$$

similarly

$$v(g'(y)) = \operatorname{Ult} S(g'(y))$$

where

$$S(g'(y)) = \{g'(x) + qg'(a_\rho)\}_{\rho \in \lambda}$$

is a pseudo-convergent sequence of M_2 with limit $g'(y) \in M_2'$.

Now $g'_{|M_1} = g$ is valuation preserving from M_1 to M_2. So we have

$$\operatorname{Ult}(S(y)) = \operatorname{Ult}(S(g'(y)))$$

hence

$$v(y) = v(g'(y))$$

as required. □

Proposition 1.4. $\mathrm{H}_{\gamma \in \Gamma} B(\gamma)$ *is pseudo-complete.*

Proof. Let $\{a_\rho\}_{\rho \in \lambda}$ be pseudo-Cauchy. Recall that

$$\gamma_\rho = v(a_\rho - a_{\rho+1})$$

is strictly increasing. Define $x \in \mathrm{H}_{\gamma \in \Gamma} B(\gamma)$ by

$$x(\gamma) = \begin{cases} a_\rho(\gamma) & \text{if } \gamma < \gamma_\rho \\ 0 & \text{otherwise.} \end{cases}$$

It is well defined because if $\rho_1 < \rho_2$, $\gamma < \gamma_{\rho_1}$ and $\gamma < \gamma_{\rho_2}$, then

$$v(a_{\rho_1} - a_{\rho_2}) = \gamma_{\rho_1}$$

and then

$$a_{\rho_1}(\gamma) = a_{\rho_2}(\gamma).$$

We show now that support(x) is well ordered.

Let $A \subseteq \text{support}(x)$, $A \neq \emptyset$ and $\gamma_0 \in A$. Then $\exists\, \rho$ such that $\gamma_0 < \gamma_\rho$ and $x(\gamma_0) = a_\rho(\gamma_0)$ with $\gamma_0 \in \text{support}(a_\rho)$.

Consider

$$A_0 := \{\gamma \in A : \gamma \leqslant \gamma_0\}.$$

Note that since $x(\gamma) = a_\rho(\gamma)$ for $\gamma \leqslant \gamma_0$ it follows that $A_0 \subseteq \text{support}(a_\rho)$ which is well ordered, so $\min A_0$ exists in A_0 and it is the least element of A.

We now conclude by showing that x is a pseudo-limit. From definition of x we have

$$v(x - a_\rho) \geqslant \gamma_\rho = v(a_{\rho+1} - a_\rho) \quad \forall\, \rho.$$

If $v(x - a_\rho) > v(a_\rho - a_{\rho+1})$, then

$$v(x - a_{\rho+1}) = v(x - a_\rho + a_\rho - a_{\rho+1}) = v(a_\rho - a_{\rho+1}) = \gamma_\rho$$

but

$$v(x - a_{\rho+1}) \geqslant \gamma_{\rho+1} > \gamma_\rho,$$

contradiction. □

As a corollary to the general embedding theorem and this proposition we get Hahn's embedding's theorem.