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INTRODUCTION TO O-MINIMAL GEOMETRY

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The aim of the seminar is to start from the beginning the study of o-minimal geometry, a very active field of research which provides a setting for generalizing semialgebraic geometry.

Reference: Lou van den Dries, **Tame topology and o-minimal structures**, LMS Lecture Notes Series, **248**, Cambridge University Press, 1998.

The following is a possible schedule with attached reference from the book mentioned above:

1. Basics. Basic algebraic and logic notions needed to the study of o-minimal structures.
Chapter 1, §1-2: (1.1) – (2.4), (2.6).

2. O-minimal structures, groups and rings. Definition of o-minimal structures. Every o-minimal ordered group is abelian, divisible and torsion-free. Every o-minimal ordered ring is a real closed field. Definition of model-theoretic structures.
Chapter 1, §3-5: (3.1) – (3.6), (4.1) – (4.6), (5.1) – (5.8).

3. The simplest one and semilinear sets. Every dense linearly ordered non-empty set without endpoints is o-minimal and it is the simplest o-minimal structure. Definition and properties of semilinear sets.
Chapter 1, §6-7: (6.1) – (6.5), (7.1) – (7.12).

4. Semialgebraic sets. Preparation to the semialgebraic cell decomposition. Proof of *Thom's lemma* and *Continuity of roots*.
Chapter 2, §1-2: (1.1) – (1.3), (2.1) – (2.6).

5. Semialgebraic cell decomposition. Proof of the cylindrical decomposition of semialgebraic sets. As a consequence one gets that the definable sets in the ordered field of real numbers are exactly the semialgebraic sets, and therefore the ordered field of real numbers is o-minimal.
Chapter 2, §2-3: (2.7) – (2.11), (3.1) – (3.6).

6. Monotonicity theorem and finiteness lemma. *Monotonicity theorem*: Every definable function over an open interval is either constant or strictly monotone and continuous on each subinterval of a finite partition of the interval. *Finiteness lemma*: If a definable set on the plane has finite fiber in each point, then there is a finite upper bound to the cardinalities of the fibers.
Chapter 3, §1: (1.1) – (1.8).

7. Cell decomposition. Proof of the fact that each definable set admits a finite partition in cells. It is the most important theorem in o-minimality.
Chapter 3, §2: (2.1) – (2.17).

8. Connected components and definable families. Each definable set has finitely many definably connected components. Definitions and properties of definable families.
Chapter 3, §2-3: (2.18), (3.1) – (3.7).

9. Dimension. Definable sets have a notion of dimension with good geometric properties, such as the dimension of the frontier of a definable set is strictly less then the dimension of that set.
Chapter 4, §1: (1.1) – (1.10).

10. Euler characteristic. Cell decomposition allows to define another invariant by definable bijections, the o-minimal Euler characteristic, which behaves on definable sets as cardinality does on finite sets.
Chapter 4, §2: (2.1) – (2.13).

11. The independence property. One of the most important dividing line between theories is the Independence Property versus the Non Independence Property. The definition here is given using the combinatorial VC-classes.
Chapter 5, §1-2: (1.1) – (2.13).

12. O-minimal structures does not have the independence property. The last lecture is devoted to the proof of it through Ramsey's theorem. Chapter 5, §3: (3.1) – (3.15).