

**REAL ALGEBRAIC GEOMETRY LECTURE NOTES**  
**(21: 29/06/15)**

SALMA KUHLMANN

CONTENTS

- |    |  |   |
|----|--|---|
| 1. | Additive lexicographic composition         | 1 |
| 2. | Multiplicative lexicographic decomposition | 2 |

**Chapter IV: Real closed exponential fields**

1. ADDITIVE LEXICOGRAPHIC COMPOSITION

**Remark 1.1.** Let  $A, B$  be ordered abelian groups. The **lexicographic product**  $A \sqcup B$  is the ordered abelian group defined as follows:

As a group it is just the direct sum  $A \oplus B$ . The total order is the lexicographic order on  $A \oplus B$ , i.e. for  $a_i \in A$  and  $b_i \in B$

$$a_1 + b_1 < a_2 + b_2 :\Leftrightarrow \text{either } a_1 < a_2 \text{ or } a_1 = a_2 \text{ and } b_1 < b_2.$$

**Recall 1.2.** A **complement**  $U$  of a subspace  $W$  of  $V$  is just a subspace such that  $V = U \oplus W$ . Moreover,  $U$  is unique up to isomorphism.

**Theorem 1.3.** *Let  $(K, +, \cdot, 0, 1, <)$  be an ordered non-Archimedean field with value group  $G$  and residue field  $\bar{K}$ . Consider the ordered divisible abelian group  $(K, +, 0, <)$ .*

- *There exists a complement  $\mathbb{A}$  of  $K_v$  in  $(K, +, 0, <)$  and a complement  $\mathbb{A}'$  of  $I_v$  in  $K_v$  such that  $(K, +, 0, <) = \mathbb{A} \sqcup \mathbb{A}' \sqcup I_v$ .*
- *Both  $\mathbb{A}$  and  $\mathbb{A}'$  are unique (up to isomorphism of ordered groups). Moreover,  $\mathbb{A}'$  is isomorphic to  $(\bar{K}, +, 0, <)$ .*
- *Furthermore the value set of  $\mathbb{A}$  is  $G^{<0}$  and the value set of  $I_v$  is  $G^{>0}$ . The Archimedean components of  $\mathbb{A}$  and  $I_v$  are all isomorphic to  $(\bar{K}, +, 0, <)$ .*

The proof of this theorem will be in the assignment. Consider

$$v : (K, +, 0, <) \rightarrow G.$$

Note that  $v(I_v) = G^{>0}$ , so  $v(\mathbb{A}) = G^{<0}$ .

**Hilfslemma 1.4.**

- (i) Let  $M$  be an ordered  $\mathbb{Q}$ -vector space and  $C$  a convex subspace of  $M$  such that  $M = C' \oplus C$ , where  $C'$  is the vector space complement of  $C$  in  $M$ . Then  $M = C' \sqcup C$ .
- (ii) Let  $\eta : M \rightarrow N$  be a surjective homomorphism of ordered vector spaces. Then  $\ker \eta$  is a convex subspace of  $M$  and  $M \cong N \sqcup \ker \eta$ .
- (iii) Let  $M, N$  be ordered vector spaces with convex subspaces  $C$  and  $D$ , respectively. Assume that  $\eta : M \rightarrow N$  is an isomorphism of ordered vector spaces such that  $\eta(C) = D$ . Then

$$\bar{\eta} : M/C \mapsto N/D, a + C \mapsto \eta(a) + D$$

is a well-defined isomorphism of ordered vector spaces.

**Remark 1.5.** Consider the divisible ordered abelian group  $(K, +, 0, <)$  and  $x = 1 \in K$ . Compute  $C_1 = (K_v, +, 0, <)$  and  $D_1 = (I_v, +, 0, <)$ . For the Archimedean component we have

$$B_1 \cong C_1/D_1 \cong (\bar{K}, +, 0, <).$$

We generalize this observation to the following:

**Proposition 1.6.** *All the Archimedean components of the divisible ordered abelian group  $(K, +, 0, <)$  are isomorphic to the divisible ordered abelian group  $(\bar{K}, +, 0, <)$ .*

*Proof.* Let  $a \in K, a > 0$ . The map

$$\eta : C_a \mapsto (\bar{K}, +, 0, <), x \mapsto \overline{xa^{-1}}$$

(Recall:  $G = \{x : v(x) \geq v(a)\}$ ) is a surjective homomorphism of ordered groups with kernel  $D_a = \{x : v(x) > v(a)\} \subset C_a$ .  $\square$

## 2. MULTIPLICATIVE LEXICOGRAPHIC DECOMPOSITION

**Theorem 2.1.** *Let  $(K, +, \cdot, 0, 1, <)$  be a totally ordered non-Archimedean field with natural valuation  $v$ ,  $G = v(K^*)$  and residue field  $\bar{K}$ . Assume that  $K$  is root closed for positive elements, i.e.  $(K^{>0}, \cdot, 1, <)$  is a divisible ordered group.*

- *There exists a group complement  $\mathbb{B}$  of  $U_v^{>0}$  in  $(K^{>0}, \cdot, 1, <)$  and a group complement  $\mathbb{B}'$  of  $1 + I_v$  in  $(U_v^{>0}, \cdot, 1, <)$  such that*

$$(K^{>0}, \cdot, 1, <) = \mathbb{B} \sqcup \mathbb{B}' \sqcup (1 + I_v, \cdot, 1, <).$$

- *Every group complement  $\mathbb{B}$  is isomorphic to  $G$ .*
- *Every group complement  $\mathbb{B}'$  is isomorphic to  $(\bar{K}^{>0}, \cdot, 1, <)$ .*

The proof follows from the following two lemmas and the Hilfslemma.

**Lemma 2.2.** *The map*

$$(K^{>0}, \cdot, 1, <) \rightarrow G, a \mapsto -v(a) = v(a^{-1})$$

*is a surjective homomorphism of ordered groups with kernel  $U_v^{>0}$ . Thus,  $U_v^{>0}$  is a convex subgroup of  $(K^{>0}, \cdot, 1, <)$  and*

$$(K^{>0}, \cdot, 1, <)/U_v^{>0} \cong G.$$

Therefore  $(K^{>0}, \cdot, 1, <) \cong \mathbb{B} \sqcup U_v^{>0}$  with  $\mathbb{B} \cong G$ .

**Lemma 2.3.** *The map*

$$(U_v^{>0}, \cdot, 1, <) \rightarrow (\overline{K}^{>0}, \cdot, 1, <), a \mapsto \bar{a},$$

*is a surjective homomorphism of ordered groups with kernel  $1 + I_v$ . Thus*

$$(U_v^{>0}, \cdot, 1, <)/(1 + I_v, \cdot, 1, <) \cong (\overline{K}^{>0}, \cdot, 1, <).$$

Therefore  $U_v^{>0} \cong \mathbb{B}' \sqcup 1 + I_v$ , where  $\mathbb{B}' \cong (\overline{K}^{>0}, \cdot, 1, <)$ .