REAL ALGEBRAIC GEOMETRY II LECTURE NOTES (02: 16/04/15 - CORRECTED ON 02/05/2019)

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Contents

1. Hahn valued modules

The **Hahn sum** is the Z-submodule of $\prod_{\gamma\in\Gamma} B(\gamma)$ consisting of all elements with finite support. We denote it by

$$\coprod_{\gamma\in\Gamma} B(\gamma) := \left\{ s \in \prod_{\gamma\in\Gamma} B(\gamma) : |\mathrm{supp}(s)| < \infty \right\}$$

We endow $\coprod_{\gamma\in\Gamma} B(\gamma)$ with the valuation

$$\begin{aligned} v_{\min}\colon \coprod_{\gamma\in\Gamma} B(\gamma) &\longrightarrow \Gamma \cup \{\infty\} \\ v_{\min}(s) &= \min \mathrm{support}(s). \end{aligned}$$

(convention: $\min \emptyset = \infty$).

The **Hahn product** is the Z-submodule of $\prod_{\gamma\in\Gamma} B(\gamma)$ consisting of all elements with well-ordered support in Γ. We denote it by

$$\mathrm{H}_{\gamma\in\Gamma} B(\gamma) := \left\{ s \in \prod_{\gamma\in\Gamma} B(\gamma) : \mathrm{supp}(s) \text{ is a well-ordered subset of } \Gamma \right\}$$

We endow $\mathrm{H}_{\gamma\in\Gamma} B(\gamma)$ with the valuation $v_{\min}$ as well.

2. Well-ordered sets

We recall that a totally ordered set Γ is **well-ordered** if every non-empty subset of Γ has a least element, or equivalently if every strictly descending sequence of elements from Γ is finite.

Example 2.1.

(1) $Z = \mathbb{Z}$, assume $\Gamma = \{1, \ldots, n\},\ B(\gamma) = \mathbb{Z}$. Then

$$\bigsqcup_{\gamma=1,\ldots,n} B(\gamma) = \mathrm{H}_{\gamma=1,\ldots,n}\, B(\gamma)$$

(2) $Z = \mathbb{Q}$, assume $\Gamma = \mathbb{N}$ (with natural order).
order type $\mathbb{N} =$ the first infinite ordinal number ω.

Let $B(\gamma) := \begin{cases} \mathbb{Q} & \text{if } \gamma \text{ is odd} \\ \mathbb{R} & \text{if } \gamma \text{ is even} \end{cases}$

Then

$$\mathrm{H}_{\gamma\in\mathbb{N}}\, B(\gamma) = \prod_{\gamma\in\mathbb{N}} B(\gamma).$$

More generally this holds whenever Γ is a well-ordered set, i.e. whenever Γ is an ordinal.

(3) $\Gamma = -\mathbb{N}$ with natural order.

$$\bigsqcup_{\gamma\in-\mathbb{N}} B(\gamma) = \mathrm{H}_{\gamma\in-\mathbb{N}}\, B(\gamma)$$

More generally this holds whenever Γ is an anti well-ordered set, i.e. well-ordered under the order relation

$$\gamma_1 \leqslant^* \gamma_2 \Leftrightarrow \gamma_2 \leqslant \gamma_1.$$

(4) $\Gamma = \mathbb{Q}$. Then

$$\bigsqcup_{\gamma\in\mathbb{Q}} B(\gamma) \subsetneq \mathrm{H}_{\gamma\in\mathbb{Q}}\, B(\gamma) \subsetneq \prod_{\gamma\in\mathbb{Q}} B(\gamma).$$

Note that every countable ordinal is the order type of a well-ordered subset of $\mathbb{Q}$.

Theorem 2.2. *(Cantor)*
Every countable dense linear order without endpoints is isomorphic to $\mathbb{Q}$.

Definition 2.3.

(i) A linear order Q is **dense** if

$$\forall q_1 < q_2 \in Q\ \exists q_3 \in Q \text{ such that } q_1 < q_3 < q_2.$$

(ii) A linear order has **no endpoints** if it has no least element and no last element.

Example 2.4.

(i) $\mathbb{Q}$ is dense because for $q_1 < q_2$ define $q_3 := \frac{q_1+q_2}{2}$.
$\mathbb{R}$ is dense.

(ii) $\mathbb{Q}$ and $\mathbb{R}$ have no endpoints.

Example 2.5.

(5) $\bigsqcup_{\gamma\in\mathbb{Q}^{<0}} B(\gamma) \subsetneq \mathrm{H}_{\gamma\in\mathbb{Q}^{<0}} B(\gamma)$

$q \mapsto -q \quad q \mapsto \frac{1}{-q}(q \neq 0) \quad 0 \mapsto 0$. Note that $\mathbb{Q}^{<0} := \{q \in \mathbb{Q} : q < 0\}$ has no endpoints.

$$\bigsqcup_{\gamma\in\mathbb{Q}^{<0}} B(\gamma) \cong \bigsqcup_{\gamma\in\mathbb{Q}} B(\gamma) \cong \bigsqcup_{\gamma\in\mathbb{Q}^{>0}} B(\gamma).$$

More generally let us now take $\Gamma = (q_1, q_2)$, the open intervall in $\mathbb{Q}$ determined by $q_1 < q_2$. Note that $(q_1, q_2) \cong \mathbb{Q}$.

(6) $\Gamma = \mathbb{R}$. Then

$$\bigsqcup_{\gamma\in\mathbb{R}} B(\gamma) \subsetneq \mathrm{H}_{\gamma\in\mathbb{R}} B(\gamma) \subsetneq \prod_{\gamma\in\mathbb{R}} B(\gamma).$$

What are the well-ordered subsets of $\mathbb{R}$?

(i) all well-ordered subsets of $\mathbb{Q}$!

(ii) all countable ordinals are the order type of some well-ordered subset of $\mathbb{R}$.

Now: Are there more?

Discussion: What is the cardinality of $\mathbb{R}$?

$$|\mathbb{R}| = \left|\{0,1\}^{\mathbb{N}}\right| = |\{0,1\}|^{\mathbb{N}} = 2^{\aleph_0} = c := \text{ the continuum}$$

Therefore

$$c = |\mathfrak{P}(\mathbb{N})| > |\mathbb{N}| = \aleph_0.$$

More precisely: are there uncountable well-ordered subsets of $\mathbb{R}$?