REAL ALGEBRAIC GEOMETRY LECTURE NOTES (06: 27/04/15 - CORRECTED ON 06/05/2019)

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Contents

1. Introduction

Our aim for this and next lecture is to complete the proof of Hahn's embedding Theorem:

Let (V, v) be a Q-valued vector space with $S(V) = [\,\Gamma, B(\gamma)\,]$.
Let $\{x_i : i \in I\} \subset V$ be maximal valuation independent and

$$h\colon V_0 = (\langle\{x_i : i \in I\}\rangle,\ v) \ \xrightarrow{\sim}\ (\bigsqcup_{\gamma\in\Gamma} B(\gamma),\ v_{\min}).$$

Then h extends to a valuation preserving embedding (i.e. an isomorphism onto a valued subspace)

$$\tilde{h}\colon (V, v) \ \hookrightarrow\ (\mathrm{H}_{\gamma\in\Gamma}\, B(\gamma),\ v_{\min}).$$

The picture is the following:

$$\begin{tikzcd}
(V,\, v) \arrow[r, hook, "\tilde{h}"] \arrow[d, dash, "\text{immediate}"'] & (\mathrm{H}_{\gamma\in\Gamma}\, B(\gamma),\ v_{\min}) \arrow[d, dash, "\text{immediate}"] \\
(V_0,\, v) \arrow[r, "h", "\sim"'] & (\bigsqcup_{\gamma\in\Gamma} B(\gamma),\ v_{\min})
\end{tikzcd}$$

2. Pseudo-convergence and maximality

Definition 2.1. A valued Q-vector space (V, v) is said to be **maximally valued** if it admits no proper immediate extension.

Definition 2.2. Let $S = \{a_\rho : \rho \in \lambda\} \subset V$ for some limit ordinal λ. Then S is said to be **pseudo-convergent** (or **pseudo-Cauchy**) if for every $\rho < \sigma < \tau$ we have

$$v(a_\sigma - a_\rho) \; < \; v(a_\tau - a_\sigma).$$

Example 2.3.

(*a*) Let $V = (\mathrm{H}_{\mathbb{N}_0}\,\mathbb{R},\, v_{\min})$, where $\mathbb{N}_0 = \{0, 1, 2, \dots\}$. An element $s \in V$ can be viewed as a function $s\colon \mathbb{N}_0 \to \mathbb{R}$. Consider

$$\begin{aligned} a_0 &= (1, 0, 0, 0, 0 \dots) \\ a_1 &= (1, 1, 0, 0, 0 \dots) \\ a_2 &= (1, 1, 1, 0, 0 \dots) \\ &\vdots \end{aligned}$$

The sequence $\{a_n : n \in \mathbb{N}_0\} \subset V$ is pseudo-Cauchy.

(*b*) Take (V, v) as above and $s \in V$ with

$$\mathrm{support}(s) = \mathbb{N}_0,$$

i.e. $s_i := s(i) \neq 0\ \forall\, i \in \mathbb{N}_0$. Define the sequence

$$\begin{aligned} b_0 &= (s_0, 0, 0, 0, 0 \dots) \\ b_1 &= (s_0, s_1, 0, 0, 0 \dots) \\ b_2 &= (s_0, s_1, s_2, 0, 0 \dots) \\ &\vdots \end{aligned}$$

For every $l < m < n \in \mathbb{N}_0$, we have

$$l + 1 = v_{\min}(b_m - b_l) < v_{\min}(b_n - b_m) = m + 1.$$

Therefore $\{b_n : n \in \mathbb{N}_0\} \subset V$ is pseudo-Cauchy.

Lemma 2.4. *If $S = \{a_\rho\}_{\rho\in\lambda}$ is pseudo-convergent then*

(*i*) *either $v(a_\rho) < v(a_\sigma)$ for all $\rho < \sigma \in \lambda$,*

(*ii*) *or $\exists\, \rho_0 \in \lambda$ such that $v(a_\rho) = v(a_\sigma)\ \ \forall\, \rho, \sigma \geqslant \rho_0$.*

Proof. Assume (*i*) does not hold, i.e. $v(a_\rho) \geqslant v(a_\sigma)$ for some $\rho < \sigma \in \lambda$. Then we claim that

$$v(a_\tau) = v(a_\sigma) \qquad \forall\, \tau > \sigma.$$

Otherwise, $v(a_\tau - a_\sigma) = \min\{v(a_\tau), v(a_\sigma)\} \leqslant v(a_\sigma)$.

But $v(a_\sigma - a_\rho) \geqslant v(a_\sigma)$, contradicting pseudo-convergence for $\rho < \sigma < \tau$. □

Notation 2.5. In case (ii) define

$$\operatorname{Ult} S := v(a_{\rho_0}) = v(a_\rho) \qquad \forall \rho \geqslant \rho_0.$$

Lemma 2.6. *If $\{a_\rho\}_{\rho\in\lambda}$ is pseudo-convergent, then for all $\rho < \sigma \in \lambda$ we have*

$$v(a_\sigma - a_\rho) = v(a_{\rho+1} - a_\rho).$$

Proof. We may assume $\sigma > \rho + 1$ (so $\rho < \rho + 1 < \sigma$). From

$$v(a_{\rho+1} - a_\rho) < v(a_\sigma - a_{\rho+1})$$

and the identity

$$a_\sigma - a_\rho = (a_\sigma - a_{\rho+1}) + (a_{\rho+1} - a_\rho),$$

we deduce that

$$\begin{aligned} v(a_\sigma - a_\rho) &= \min\{v(a_\sigma - a_{\rho+1}), v(a_{\rho+1} - a_\rho)\} \\ &= v(a_{\rho+1} - a_\rho). \end{aligned}$$

□

Notation 2.7.

$$\begin{aligned} \gamma_\rho :&= v(a_{\rho+1} - a_\rho) \\ &= v(a_\sigma - a_\rho) \qquad \forall\, \sigma > \rho. \end{aligned}$$

Remark 2.8. Since $\rho < \rho + 1 < \rho + 2$, we have $\gamma_\rho < \gamma_{\rho+1}$ for all $\rho \in \lambda$.

3. Pseudo-limits

Definition 3.1. Let $S = \{a_\rho\}_{\rho\in\lambda}$ be a pseudo-convergent set. We say that $x \in V$ is a **pseudo-limit** of S if

$$v(x - a_\rho) = \gamma_\rho \qquad \text{for all } \rho \in \lambda.$$

Remark 3.2.

(i) If $v(a_\rho) < v(a_\sigma)$ for $\rho < \sigma$, then $x = 0$ is a pseudo-limit.

(ii) If 0 is not a pseudo-limit and x is a pseudo-limit, then $v(x) = \operatorname{Ult} S$.

Example 3.3.

(a) In Example 2.3(a), the constant function 1:

$$a = (1, 1, \dots)$$

is a pseudo-limit of the sequence $\{a_n\}_{n\in\mathbb{N}_0}$.

(*b*) In Example 2.3(*b*), s is a pseudo-limit of $\{b_n\}_{n\in\mathbb{N}_0}$.

Definition 3.4. (V, v) is **pseudo-complete** if every pseudo-convergent sequence in V has a pseudo-limit in V.

We will analyse the set of pseudo-limits of a given pseudo-Cauchy sequence (this set can be empty, a singleton, or infinite).

Definition 3.5. Let $S = \{a_\rho\}_{\rho\in\lambda}$ be a pseudo-convergent set. The **breadth** (*Breite*) B of S is defined to be the following subset of V:

$$B = B(S) := \{y \in V : v(y) > \gamma_\rho \ \forall\, \rho \in \lambda\}.$$

Lemma 3.6. *Let $S = \{a_\rho\}_{\rho\in\lambda}$ be pseudo-convergent with breadth B and let $x \in V$ be a pseudo-limit of S. Then an element of V is a pseudo-limit of S if and only if it is of the form $x + y$ with $y \in B$.*

Proof.

($\Rightarrow$) Let z be another pseudo-limit of S. It follows from

$$x - z = (x - a_\rho) - (z - a_\rho)$$

that

$$v(x - z) \geqslant \min\{v(x - a_\rho), v(z - a_\rho)\} = \gamma_\rho \qquad \forall\, \rho \in \lambda.$$

Since γ_ρ is strictly increasing, it follows $v(x-z) > \gamma_\rho$ for all $\rho \in \lambda$. So $x - z \in B$ is as required.

($\Leftarrow$) If $y \in B$ then $v(y) > \gamma_\rho = v(x - a_\rho)$ for all $\rho \in \lambda$. Then

$$v((x + y) - a_\rho) = v((x - a_\rho) + y) = \min\{v(x - a_\rho), v(y)\} = \gamma_\rho \qquad \forall\, \rho \in \lambda.$$

□

4. Cofinality

Definition 4.1. Let Γ be a totally ordered set. A subset $A \subset \Gamma$ is cofinal in Γ if

$$\forall\, \gamma \in \Gamma \ \exists\, a \in A \text{ with } \gamma \leqslant a.$$

Example 4.2. If $\Gamma = [0, 1] \subset \mathbb{R}$, then $A = \{1\}$ is cofinal in Γ.

Lemma 4.3. *Let $\emptyset \neq \Gamma$ be a totally ordered set. Then there is a well-ordered cofinal subset $A \subset \Gamma$. Moreover if Γ has no last element, then A has also no last element, i.e. the order type of A is a limit ordinal.*

Remark 4.4. Note that if $\{a_\rho\}_{\rho\in\lambda}$ is pseudo-Cauchy in (V, v), $x \in V$ is a pseudo-limit and $\{\gamma_\rho\}_{\rho\in\lambda}$ is cofinal in $\Gamma = v(V)$, then it follows by Lemma 3.6 that the limit is unique. This is because if $\{\gamma_\rho\}_{\rho\in\lambda}$ is cofinal in Γ, then $B(S) = \{0\}$.

Warning: $\{\gamma_\rho\}_{\rho\in\lambda}$ is cofinal in $\Gamma \not\Rightarrow S$ has no limit.