REAL ALGEBRAIC GEOMETRY LECTURE NOTES
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1. The rank of ordered fields

(Applications later on: the rank of a Hardy-field).

Definition 1.1. Let K be a field and w and w' be valuations on K. We say that w' is **finer** than w or that w is **coarser** than w', if $K_{w'} \subseteq K_w$ (or equivalently $I_w \subseteq I_{w'}$).

Remark 1.2.

(*i*) An overring of a valuation ring is a valuation ring.

(*ii*) If w' is a convex valuation and w is coarser than w', then w is a convex valuation.

(*iii*) We have proved that the natural valuation on an ordered field K induces the smallest (for inclusion) convex valuation ring of K.

(*iv*) The collection of all convex valuations (respectively valuation rings) of K is totally ordered by inclusion.

Definition 1.3. The **rank** of the totally ordered field K is the (order type of the totally ordered) set

$$\mathcal{R} := \{K_w : K_w \text{ is a convex valuation and } K_v \subsetneq K_w\},$$

where v denotes the natural valuation. Note that

$$\mathcal{R} := \{K_w : w \text{ is strictly coarser than } v\}.$$

Example 1.4.

- The rank of an Archimedean ordered field is empty (since its natural valuation is trivial), its order type 0.

- The rank of the rational function field $K = \mathbb{R}(t)$ with any order is a singleton. Indeed the field $\mathbb{R}(t)$ is non-Archimedean under any order (see RAG I). Moreover, any ordering of $\mathbb{R}(t)$ has rank 1.

2. The Descent

From the ordered field K down to the ordered group $v(K^\times) =: G$.

Let K_w be a convex valuation ring of K. We associate to w the following subset of G :

$$\begin{aligned} G_w &:= \{v(a) : a \in K,\, w(a) = 0\} \\ &= \{v(a) : a \in K^{>0},\, w(a) = 0\} \\ &= v(U_w) = v(U_w^{>0}). \end{aligned}$$

Remark 2.1. Note that w is a coarsening of v if the following holds:

$$v(a) \leqslant v(b) \Rightarrow w(a) \leqslant w(b).$$

Lemma 2.2. *G_w is a convex subgroup of G.*

Proof.

- $0 = v(1)$ and $1 \in U_w$.
- Let $g \in G_w$. Show $-g \in G_w$. Let $a \in U_w$ such that $g = v(a)$. Then $a^{-1} \in U_w$ and
$$G_w \ni v(a^{-1}) = -v(a) = -g.$$
- Similarly assume $g_1, g_2 \in G_w$. There exist $a_1, a_2 \in U_w$ such that $v(a_i) = g_i$. Then $a_1 a_2 \in U_w$ and
$$v(a_1 a_2) = v(a_1) + v(a_2) = g_1 + g_2 \in G_w.$$
- Let $g \in G_w$ and $0 < h < g$ for some $h \in G$. Show $h \in G_w^{>0}$. Let $g = v(b), b \in U_w$, and $h = v(a)$ for some $a \in K^{>0}$. Then
$$v(a) \leqslant v(b) \Rightarrow w(a) \leqslant w(b) = 0 \Rightarrow w(a) = 0.$$

□

Lemma 2.3. *The value group $w(K^\times)$ is isomorphic (as an ordered group) to $v(K^\times)/G_w$, so*

$$w(K^\times) \cong v(K^\times)/v(U_w).$$

Proof. Consider the map

$$\phi : v(K^\times) \to w(K^\times),\; v(a) \mapsto w(a).$$

Compute

$$\begin{aligned} \ker\phi &= \{v(a) : \phi(v(a)) = 0\} \\ &= \{v(a) : w(a) = 0\} \\ &= G_w, \end{aligned}$$

i.e. ϕ is a surjective homomorphism with kernel G_w, so $w(K^\times) \cong v(K^\times)/G_w$. Moreover this isomorphism is order preserving: note that since G_w is a convex subgroup of $v(K^\times)$, the group $v(K^\times)/G_w$ is totally ordered.

□

Definition 2.4. Given w a coarsening of v, we call $G_w = v(U_w)$ the **convex subgroup of G associated to w.**

Conversely, we get the following result:

Lemma 2.5. *Given any convex subgroup C of G we define a valuation w on K as follows:*

$$w : K^\times \to v(K^\times)/C,\ w(a) = v(a) + C \quad \textit{(the canonical map)}$$

Then w is a convex valuation on K and $G_w = C$.

Proof.

- $v(a) \in G_w \Leftrightarrow w(a) = 0 \Leftrightarrow v(a) \in C$.
- $$\begin{aligned} w(a+b) = v(a+b) + C &\geqslant \min\{v(a) + C, v(b) + C\} \\ &\Leftrightarrow v(a+b) \geqslant \min\{v(a), v(b)\} \\ &\Leftrightarrow w(a+b) \geqslant \min\{w(a), w(b)\}. \end{aligned}$$
- $0 < a \leqslant b \Rightarrow v(a) \geqslant v(b) \Rightarrow v(a) + C \geqslant v(b) + C \Rightarrow w(a) \geqslant w(b)$.
- $$\begin{aligned} w(ab) = v(ab) + C &= (v(a) + v(b)) + C \\ &= (v(a) + C) + (v(b) + C) \\ &= w(a) + w(b). \end{aligned}$$

□

Definition 2.6. w is called the **convex valuation associated to C.**

Let us summarize:

Proposition 2.7. *Suppose that w is coarser than v. Then for all $a, b \in K$:*

$$v(a) \leqslant v(b) \Rightarrow w(a) \leqslant w(b).$$

Let $G_w = v(U_w)$ be the convex subgroup of $v(K^\times)$ associated to w. Then

$$w(K^\times) \cong v(K^\times)/G_w.$$

Conversely every convex subgroup C of $v(K^\times)$ is of the form G_w, where w is the convex valuation associated to C.

Corollary 2.8. *(Descent into the value group)*
The correspondence $K_w \mapsto G_w$ is a one to one (inclusion) order preserving correspondence between the rank of K and the rank of $G = v(K^\times)$.

Example 2.9.

(*i*) $K = \mathbb{R}((\mathbb{Z}))$ the field of Laurent series ordered lex. Then $\mathcal{R}_K = 1$.

(*ii*) $K = \mathbb{R}((\mathbb{Q})) \Rightarrow$ rank is 1,

(*iii*) $K = \mathbb{R}((\mathbb{R})) \Rightarrow$ rank is 1.

(*iv*) $K = \mathbb{R}((\mathbb{Z} \times \mathbb{Z})) \Rightarrow$ rank is 2.