REAL ALGEBRAIC GEOMETRY LECTURE NOTES (24: 09/07/15 - CORRECTED ON 09/07/19)

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1. Baer-Krull Representation Theorem 1

1. Baer-Krull Representation Theorem

Recall that an ordering $\leqslant$ and a valuation v on a field K are called compatible if

$$0 \leqslant x \leqslant y \Rightarrow v(y) \leqslant v(x).$$

In Proposition 2.3 of lecture 17 we fixed an ordered field $(K, \leqslant)$ and characterized the $\leqslant$-compatible valuations on K. Today, we fix a valued field (K, v) and describe the v-compatible orderings on K.

Notation 1.1. Let (K, v) be a valued field. Let Γ be the value group of v. The quotient group $\overline{\Gamma} = \Gamma/2\Gamma$ becomes in a canonical way an $\mathbb{F}_2$-vector space. We denote by $\overline{\gamma} = \gamma + 2\Gamma$ the residue class of $\gamma \in \Gamma$.
Let $\{\pi_i : i \in I\} \subseteq K^*$ such that $\{\overline{v(\pi_i)} : i \in I\}$ is an $\mathbb{F}_2$-basis of $\overline{\Gamma}$. Then $\{\pi_i : i \in I\}$ is called a **quadratic system of representatives** of K with respect to v.

Theorem 1.2. *(Baer-Krull Representation Theorem)*
Let (K, v) be a valued field. Let $\mathcal{X}(K)$ and $\mathcal{X}(Kv)$ denote the set of all orderings of K and Kv, respectively. Fix some quadratic system $\{\pi_i : i \in I\}$ of representatives of K with respect to v.
Then there is a bijective correspondence

$$\{P \in \mathcal{X}(K) : K_v \text{ is } P\text{-convex}\} \longleftrightarrow \{-1, 1\}^I \times \mathcal{X}(Kv)$$

described as follows: given an ordering P on K such that K_v is P-convex, let $\eta_P : I \to \{-1, 1\}$, where $\eta_P(i) = 1 \Leftrightarrow \pi_i \in P$. Then the map

$$P \mapsto (\eta_P, \overline{P})$$

is the above bijective correspondence.

Proof. Given a mapping $\eta : I \to \{-1, 1\}$ and an ordering Q on Kv, we will define an ordering $P(\eta, Q)$ on K, such that K_v is $P(\eta, Q)$-convex and $P(\eta, Q)$ is mapped to (η, Q) by the map described in the claim.
Let $a \in K^*$. As $\{\overline{v(\pi_i)} : i \in I\}$ is a basis of $\overline{\Gamma}$, there exist uniquely determined indices $i_1, \ldots, i_r$ such that

$$\overline{v(a)} = \overline{v(\pi_{i_1})} + \ldots + \overline{v(\pi_{i_r})}.$$

Thus, for some $b \in K$, one has

$$\begin{aligned} v(a) &= v(\pi_{i_1}) + \ldots + v(\pi_{i_r}) + 2v(b) \\ &= v(\pi_{i_1} \ldots \pi_{i_r} b^2). \end{aligned}$$

Hence, we find some $u \in U_v$ such that

$$a = u\pi_{i_1} \cdots \pi_{i_r} b^2.$$

Note that since b is only determined up to a unit, u is only determined up to a unit square. Let $\eta : I \to \{-1, 1\}$ be a mapping and $Q \in \mathcal{X}(Kv)$ a positive cone on Kv. We define $P(\eta, Q) \subset K$ by $0 \in P(\eta, Q)$ and for each $a \in K^*$ with $a = u\pi_{i_1} \cdot \ldots \cdot \pi_{i_r} b^2$ as above,

$$a \in P(\eta, Q) :\Leftrightarrow \eta(i_1) \cdots \eta(i_r)\overline{u} \in Q.$$

Note that $P(\eta, Q)$ is well-defined, as u and hence $\overline{u}$ is determined up to a unit square and $i_1, \ldots, i_r$ are completely determined. We have to show that $P(\eta, Q)$ is an ordering such that K_v is $P(\eta, Q)$-convex, and that $\overline{P(\eta, Q)} = Q$. We first show that $P(\eta, Q)$ is additively closed. Let $a, a' \in P(\eta, Q)$ with $a, a' \neq 0$. Moreover, let $u, u' \in U_v$, $b, b' \in K$ and $i_1, \ldots, i_r, j_1, \ldots, j_s \in I$ such that

$$\begin{aligned} a &= u\pi_{i_1} \cdots \pi_{i_r} b^2, \\ a' &= u'\pi_{j_1} \cdots \pi_{j_s} (b')^2. \end{aligned}$$

If $v(a) \neq v(a')$, say $v(a) < v(a')$, then $v(a + a') = v(a)$. Hence, $a + a' = ca$ for some $c \in U_v$. Note that $\frac{a'}{a} \in I_v$. Thus, from $1 + \frac{a'}{a} = c$ follows $\overline{c} = \overline{1}$. We obtain $a + a' = cu\pi_{i_1} \cdots \pi_{i_r} b^2$.
As $a \in P(\eta, Q)$ we have

$$\begin{aligned} Q \ni \eta(i_1) \cdots \eta(i_r)\overline{u} &= \eta(i_1) \cdots \eta(i_r)\overline{1}\overline{u} \\ &= \eta(i_1) \cdots \eta(i_r)\overline{cu}. \end{aligned}$$

Hence, $a + a' \in P(\eta, Q)$.
If $v(a) = v(a')$, then $\{i_1, \ldots, i_r\} = \{j_1, \ldots, j_s\}$. Furthermore, $b' = bu''$ for some $u'' \in U_v$. Hence,

$$a + a' = (u + u'(u'')^2)b^2\pi_{i_1} \cdots \pi_{i_r}.$$

If $\eta(\pi_{i_1}) \cdots \eta(\pi_{i_r}) = 1$, then $\overline{u}, \overline{u'} \in Q$ and hence

$$\overline{u + u' + (u'')^2} = \eta(\pi_{i_1}) \cdots \eta(\pi_{i_r})\overline{u + u' + (u'')^2} \in Q,$$

i.e. $a + a' \in P(\eta, Q)$.
If $\eta(\pi_{i_1}) \cdots \eta(\pi_{i_r}) = -1$, then $-\overline{u}, -\overline{u'} \in Q$. Hence

$$-\overline{u + u'(u'')^2} = \eta(\pi_{i_1}) \cdots \eta(\pi_{i_r})\overline{u + u'(u'')^2} \in Q,$$

and therefore $a + a' \in P(\eta, Q)$.
In order to prove that $P(\eta, Q)$ is closed under multiplication, we extend η to an $\mathbb{F}_2$-linear map from $\overline{\Gamma}$ to $\{-1, 1\}$. We define $\eta(\overline{v(\pi_i)}) = \eta(i)$, which determines the map completely, since the elements $\overline{v(\pi_i)}$ form a basis for $\overline{\Gamma}$. By composing

$$K^* \xrightarrow{v} \Gamma \to \overline{\Gamma} \xrightarrow{\eta} \{-1, 1\}$$

we obtain a group homomorphism $K^* \to \{-1, 1\}$, which we also denote by η. We have $a \in P(\eta, Q)$ if and only if $\eta(a)\overline{u} \in Q$ for all $a \in K^*$. From this it

follows at once that $aa' \in P(\eta, Q)$ for $a, a' \in P(\eta, Q)$.
As $Q \cup -Q = Kv$, it is clear from the definition that

$$P(\eta, Q) \cup -P(\eta, Q) = K$$

and as $-1 \notin Q$, it is clear that $-1 \notin P(\eta, Q)$.
Further note that $1 + I_v \subseteq P(\eta, Q)$. Indeed, if $x \in I_v$, then $v(1+x) = 0$, i.e. $1 + x = ub^2$. Thus,

$$\overline{ub^2} = \overline{1+x} = \overline{1} \in Q,$$

which implies that $1 + x \in P(\eta, Q)$. Hence, by Proposition 2.3 of lecture 17, K_v is $P(\eta, Q)$-convex. This shows that $P(\eta, Q)$ is a positive cone of K and that K_v is $P(\eta, Q)$-convex.
We still have to prove that the mapping from the claim is bijective. Let $u \in U_v \cap P(\eta, Q)$. Then it follows from the definition that $\overline{u} \in Q$. Hence, $\overline{P(\eta, Q)} \subseteq Q$. As $\overline{P(\eta, Q)}$ and Q are both positive cones, $\overline{P(\eta, Q)} = Q$. Moreover, $\pi_i \in P(\eta, Q) \Leftrightarrow \eta(\pi_i) = 1$ is clear from the definition. This proves surjectivity of the map described in the claim.
Injectivity: Assume P is mapped to (η, Q). It is clear from the definition that $P(\eta, Q) \subseteq P$, and threfore $P(\eta, Q) = P$.

□

Remark 1.3. Under additional assumptions, either factor of the cartesian product in the Baer-Krull Theorem may vanish.

(1) If Γ is 2-divisible, then $\overline{\Gamma} = \{0\}$, and therefore $I = \emptyset$. Thus, there is a bijective correspondence

$$\{P \in \mathcal{X}(K) : K_v \text{ is } P\text{-convex}\} \longleftrightarrow \mathcal{X}(Kv)$$

(2) If $\sum (Kv)^2$ is an ordering, then Kv is uniquely ordered (see RAG I). Thus, there is a bijective correspondence

$$\{P \in \mathcal{X}(K) : K_v \text{ is } P\text{-convex}\} \longleftrightarrow \{-1, 1\}^I.$$

Remark 1.4. If $(K, \leqslant)$ is an ordered field, then

$$\mathbb{Z}(\leqslant) := \{x \in K : x, -x \leqslant a \text{ for some } a \in \mathbb{Z}\}$$

is called the **$\leqslant$-convex hull of $\mathbb{Z}$** in K. It is a valuation ring on K which is non-trivial (i.e. $\neq K$) if and only if $\leqslant$ is non-Archimedean.

Corollary 1.5. *A field K admits a non-Archimedean ordering if and only if K carries a non-trivial valuation with real residue class field.*

Proof. Let P be a non-Archimedean ordering on K. Then $\mathbb{Z}(P)$ corresponds to a non-trivial valuation v on K, and $K_v = \mathbb{Z}(P)$ is P-convex. Applying the Baer-Krull Theorem to (K, v) yields that P corresponds to $(\eta_P, \overline{P})$. In particular, $\overline{P}$ is an ordering on Kv, i.e. Kv is real.
Conversely, let v be a non-trivial valuation on K (i.e. $K_v \subsetneq K$) with real residue class field Kv. Let Q be an ordering on Kv and choose $\eta = 1$ (i.e. $\eta(i) = 1$ for all i). By the Baer-Krull Theorem, there exists an ordering P of K for which K_v is P-convex. Note that $\mathbb{Z}(P) \subseteq K_v \subsetneq K$, since $\mathbb{Z}(P)$ is the smallest P-convex subring of K. Thus, P is non-Archimedean. □