

Reelle algebraische Geometrie I  
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**REAL ALGEBRAIC GEOMETRY LECTURE NOTES**  
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**Convention:** When a new definition is given, the German name appears between brackets.

1. ORDERINGS

**Definition 1.1.** (*partielle Anordnung*) Let  $\Gamma$  be a non-empty set and let  $\leq$  be a relation on  $\Gamma$  such that:

$$(i) \quad \gamma \leq \gamma \quad \forall \gamma \in \Gamma,$$

$$(ii) \quad \gamma_1 \leq \gamma_2, \gamma_2 \leq \gamma_1 \Rightarrow \gamma_1 = \gamma_2 \quad \forall \gamma_1, \gamma_2 \in \Gamma,$$

$$(iii) \quad \gamma_1 \leq \gamma_2, \gamma_2 \leq \gamma_3 \Rightarrow \gamma_1 \leq \gamma_3 \quad \forall \gamma_1, \gamma_2, \gamma_3 \in \Gamma.$$

Then  $\leq$  is a **partial order** on  $\Gamma$  and  $(\Gamma, \leq)$  is said to be a **partially ordered set**.

**Example 1.2.** Let  $X$  be a non-empty set. For every  $A, B \subseteq X$ , the relation

$$A \leq B \iff A \subseteq B,$$

is a partial order on the power set  $\mathcal{P}(X) = \{A : A \subseteq X\}$ .

**Definition 1.3.** (*totale Anordnung*) A partial order  $\leq$  on a set  $\Gamma$  is said to be **total** if

$$\forall \gamma_1, \gamma_2 \in \Gamma \quad \gamma_1 \leq \gamma_2 \text{ or } \gamma_2 \leq \gamma_1.$$

**Notation 1.4.** If  $(\Gamma, \leq)$  is a partially ordered set and  $\gamma_1, \gamma_2 \in \Gamma$ , then we write:

$$\gamma_1 < \gamma_2 \iff \gamma_1 \leq \gamma_2 \text{ and } \gamma_1 \neq \gamma_2,$$

$$\gamma_1 \geq \gamma_2 \iff \gamma_2 \leq \gamma_1,$$

$$\gamma_1 > \gamma_2 \iff \gamma_2 \leq \gamma_1 \text{ and } \gamma_1 \neq \gamma_2.$$

**Examples 1.5.** Let  $\Gamma = \mathbb{R} \times \mathbb{R} = \{(a, b) : a, b \in \mathbb{R}\}$ .

(1) For every  $(a_1, b_1), (a_2, b_2) \in \mathbb{R} \times \mathbb{R}$  we can define

$$(a_1, b_1) \leq (a_2, b_2) \iff a_1 \leq a_2 \text{ and } b_1 \leq b_2.$$

Then  $(\mathbb{R} \times \mathbb{R}, \leq)$  is a partially ordered set.

(2) For every  $(a_1, b_1), (a_2, b_2) \in \mathbb{R} \times \mathbb{R}$  we can define

$$(a_1, b_1) \leq_l (a_2, b_2) \iff [a_1 < a_2] \text{ or } [a_1 = a_2 \text{ and } b_1 \leq b_2].$$

Then  $(\mathbb{R} \times \mathbb{R}, \leq_l)$  is a totally ordered set. (Remark: the "l" stands for "lexicographic").

## 2. ORDERED FIELDS

**Definition 2.1.** (*angeordneter Körper*) Let  $K$  be a field. Let  $\leq$  be a total order on  $K$  such that:

$$(i) \quad x \leq y \Rightarrow x + z \leq y + z \quad \forall x, y, z \in K,$$

$$(ii) \quad 0 \leq x, 0 \leq y \Rightarrow 0 \leq xy \quad \forall x, y \in K.$$

Then the pair  $(K, \leq)$  is said to be an **ordered field**.

**Examples 2.2.** The field of the rational numbers  $(\mathbb{Q}, \leq)$  and the field of the real numbers  $(\mathbb{R}, \leq)$  are ordered fields, where  $\leq$  denotes the usual order.

**Definition 2.3.** (*formal reell Körper*) A field  $K$  is said to be **(formal) real** if there is an order  $\leq$  on  $K$  such that  $(K, \leq)$  is an ordered field.

**Proposition 2.4.** Let  $(K, \leq)$  be an ordered field. The following hold:

- $a \leq b \iff 0 \leq b - a \quad \forall a, b \in K$
- $0 \leq a^2 \quad \forall a \in K$
- $a \leq b, 0 \leq c \Rightarrow ac \leq bc \quad \forall a, b, c \in K$
- $0 < a \leq b \Rightarrow 0 < 1/b \leq 1/a \quad \forall a, b \in K$
- $0 < n \quad \forall n \in \mathbb{N}$

**Remark 2.5.** If  $K$  is a real field then  $\text{char}(K) = 0$  and  $K$  contains a copy of  $\mathbb{Q}$ .

**Notation 2.6.** Let  $(K, \leq)$  be an ordered field and let  $a \in K$ .

$$\text{sign}(a) := \begin{cases} 1 & \text{if } a > 0, \\ 0 & \text{if } a = 0, \\ -1 & \text{if } a < 0. \end{cases}$$

$$|a| := \text{sign}(a)a.$$

**Fact 2.7.** Let  $(K, \leq)$  be an ordered field and let  $a, b \in K$ . Then

$$(i) \text{ sign}(ab) = \text{sign}(a) \text{sign}(b),$$

$$(ii) |ab| = |a||b|,$$

$$(iii) |a + b| \leq |a| + |b|.$$

### 3. ARCHIMEDEAN FIELDS

**Definition 3.1.** (*archimedischer Körper*) Let  $(K, \leq)$  be a field. We say that  $K$  is **Archimedean** if

$$\forall a \in K \exists n \in \mathbb{N} \text{ such that } a < n.$$

**Definition 3.2.** Let  $(\Gamma, \leq)$  be an ordered set and let  $\Delta \subseteq \Gamma$ . Then

- $\Delta$  is **cofinal** (*kofinal*) in  $\Gamma$  if

$$\forall \gamma \in \Gamma \exists \delta \in \Delta \text{ such that } \gamma \leq \delta.$$

- $\Delta$  is **coinitial** (*koinitial*) in  $\Gamma$  if

$$\forall \gamma \in \Gamma \exists \delta \in \Delta \text{ such that } \delta \leq \gamma.$$

- $\Delta$  is **coterminal** (*koterminal*) in  $\Gamma$  if  $\Delta$  is cofinal and coinitial in  $\Gamma$ .

**Example 3.3.** Let  $(K, \leq)$  be an Archimedean field. Then  $\mathbb{N}$  is cofinal in  $K$ ,  $-\mathbb{N}$  is coinitial in  $K$  and  $\mathbb{Z} = -\mathbb{N} \cup \mathbb{N}$  is coterminal in  $K$ .

**Remark 3.4.**

- If  $(K, \leq)$  is an Archimedean field and  $Q \subseteq K$  is a subfield, then  $(Q, \leq)$  is an Archimedean field.
- $(\mathbb{R}, \leq)$  is an Archimedean field and therefore also  $(\mathbb{Q}, \leq)$  is.

**Remark 3.5.** Let  $(K, \leq)$  be an ordered field. Then  $K$  is Archimedean if and only if  $\forall a, b \in K^* \exists n \in \mathbb{N}$  such that

$$|a| \leq n|b| \text{ and } |b| \leq n|a|.$$

**Example 3.6.** Let  $\mathbb{R}[x]$  be the ring of the polynomials with coefficients in  $\mathbb{R}$ . We denote by  $ff(\mathbb{R}[x])$  the field of the rational functions of  $\mathbb{R}[x]$ , i.e.

$$ff(\mathbb{R}[x]) = \mathbb{R}(x) := \left\{ \frac{f(x)}{g(x)} : f(x), g(x) \in \mathbb{R}[x] \text{ and } g(x) \neq 0 \right\}.$$

Let  $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \in \mathbb{R}[x]$  and let  $k \in \mathbb{N}$  the smallest index such that  $a_k \neq 0$  (and therefore actually  $f(x) = a_n x^n + \cdots + a_k x^k$ ). We define

$$f(x) > 0 \Leftrightarrow a_k > 0$$

and then for every  $f(x), g(x) \in \mathbb{R}[x]$  with  $g(x) \neq 0$  we define

$$\frac{f(x)}{g(x)} \geq 0 \Leftrightarrow f(x)g(x) \geq 0.$$

This is a total order on  $K = f f(\mathbb{R}[x])$  which makes  $(K, \leq)$  an ordered field. We claim that  $(K, \leq)$  contains

(i) an infinite positive element, i.e.

$$\exists A \in K \text{ such that } A > n \quad \forall n \in \mathbb{N},$$

(ii) an infinitesimal positive element, i.e.

$$\exists a \in K \text{ such that } 0 < a < 1/n \quad \forall n \in \mathbb{N}.$$

For instance the element  $x \in K$  is infinitesimal and the element  $1/x \in K$  is infinite. Therefore  $(K, \leq)$  is not Archimedean.