REAL ALGEBRAIC GEOMETRY LECTURE NOTES (11: 24/11/09 - BEARBEITET 29/11/18)

SALMA KUHLMANN

Contents

1. Normal form of semialgebraic sets

Let R be a fixed real closed field and $n \geqslant 1$. We consider 3 operations on subsets of R^n:

(1) finite unions,
(2) finite intersections,
(3) complements.

Definition 1.1.

(i) The class of **semialgebraic sets** in R^n is defined to be the smallest class of subsets of R^n closed under operations (1), (2), (3), and which contains all sets of the form

$$\{\underline{x} \in R^n : f(\underline{x}) \lhd 0\},$$

where $f \in R[\underline{x}] = R[x_1, \dots, x_n]$ and $\lhd \in \{\geqslant, >, =, \neq\}$.

(ii) Equivalently a subset $S \subseteq R^n$ is semialgebraic if and only if it is a finite boolean combination of sets of the form

$$\{\underline{x} \in R^n : f(\underline{x}) > 0\},$$

where $f(\underline{x}) \in R[\underline{x}]$.

(iii) Consider

$$(*) \quad S(\underline{x}) := \begin{cases} f_1(\underline{x}) \lhd_1 0 \\ \quad \vdots \\ f_k(\underline{x}) \lhd_k 0 \end{cases}$$

with $f_i(\underline{x}) \in R[\underline{x}]$; $\lhd_i \in \{\geqslant, >, =, \neq\}$.
The set of solutions of $S(\underline{x})$ is precisely the semialgebraic set

$$S := \bigcap_{i=1}^{k} \{\underline{x} \in R^n : f_i(\underline{x}) \lhd_i 0\}.$$

The solution set S of a system $(*)$ is called a **basic semialgebraic subset** of R^n.

(*iv*) Let $f_1, \dots, f_k \in R[\underline{x}] = R[x_1, \dots, x_n]$. A set of the form

$$Z(f_1, \dots, f_k) := \{\underline{x} \in R^n : f_1(\underline{x}) = \cdots = f_k(\underline{x}) = 0\}$$

is called an **algebraic set**.

(*v*) A subset of R^n of the form

$$\begin{aligned} \mathcal{U}(f) &:= \{\underline{x} \in R^n : f(\underline{x}) > 0\}, \\ \mathcal{U}(f_1, \dots, f_k) &:= \{\underline{x} \in R^n : f_1(\underline{x}) > 0, \dots, f_k(\underline{x}) > 0\} \\ &= \mathcal{U}(f_1) \cap \cdots \cap \mathcal{U}(f_k) \end{aligned}$$

is called a **basic open semialgebraic set**.

(*vi*) A subset of R^n of the form

$$\begin{aligned} \mathcal{K}(f) &:= \{\underline{x} \in R^n : f(\underline{x}) \geqslant 0\}, \\ \mathcal{K}(f_1, \dots, f_k) &= \mathcal{K}(f_1) \cap \cdots \cap \mathcal{K}(f_k) \end{aligned}$$

is called a **basic closed semialgebraic set**.

Remark 1.2.

(*a*) An algebraic set is in particular a basic semialgebraic set.
(*b*) $Z(f_1, \dots, f_k) = Z(f)$, where $f = \sum_{i=1}^{k} f_i^2$.

Proposition 1.3.

(1) *A subset of R^n is semialgebraic if and only if it is a finite union of basic semialgebraic sets.*

(2) *A subset is semialgebraic if and only if it is a finite union of basic semialgebraic sets of the form*

$$Z(f) \cap \mathcal{U}(f_1, \dots, f_k)$$

(normal form).

Proof. (1) ((2) is similar).

($\Leftarrow$) Clear.
($\Rightarrow$) To show that the class of semialgebraic sets is included in the class of finite unions of basic semialgebraic sets it suffices to show that this last class is closed under finitary boolean operations: union, intersection, complement.

The closure by union is by definition.

Intersection:

$$(\cup_i C_i) \cap (\cup_j D_j) = \cup_{i,j} (C_i \cap D_j).$$

Complement: It is enough to show that the complement of

$$\{\underline{x} \in R^n : f(\underline{x}) \lhd 0\} \quad \lhd \in \{\geqslant, >, =, \neq\},$$

is a finite union of basic semialgebraic, since

$$(C \cap D)^c = C^c \cup D^c \quad \text{and} \quad (C \cup D)^c = C^c \cap D^c.$$

Let us consider the possible cases for $\lhd \in \{\geqslant, >, =, \neq\}$:

$\{\underline{x} \in R^n : f(\underline{x}) \geqslant 0\}^c = \{\underline{x} \in R^n : -f(\underline{x}) > 0\}$

$\{\underline{x} \in R^n : f(\underline{x}) > 0\}^c = \{\underline{x} \in R^n : f(\underline{x}) = 0\} \cup \{\underline{x} \in R^n : -f(\underline{x}) > 0\}$

$\{\underline{x} \in R^n : f(\underline{x}) = 0\}^c = \{\underline{x} \in R^n : f(\underline{x}) \neq 0\}.$

□

2. Geometric version of Tarski-Seidenberg

We shall return to a systematic study of the class of semialgebraic sets and its property in the next lectures.

For now we want to derive an important property of this class from Tarski-Seidenberg's theorem:

Theorem 2.1. *(Tarski-Seidenberg geometric version)*
Consider the projection map

$$\begin{aligned} \pi\colon R^{m+n} = R^m \times R^n &\longrightarrow R^m \\ (\underline{t},\ \underline{x}) &\mapsto \underline{t}. \end{aligned}$$

Then for any semialgebraic set $A \subseteq R^{m+n}$, $\pi(A)$ is a semialgebraic set in R^m.

Proof. Since

$$\pi(\bigcup_i A_i) = \bigcup_i \pi(A_i),$$

it suffices to show the result for a basic semialgebraic subset A of R^{m+n}; i.e. show that $\pi(A)$ is semialgebraic in R^m.

Let $\underline{u} := (u_1, \ldots, u_q)$ be the coefficients of all polynomials $f_1(\underline{T}, \underline{X}), \ldots, f_k(\underline{T}, \underline{X}) \in R[T_1, \ldots, T_m, X_1, \ldots, X_n]$ of the system $S(\underline{T}, \underline{X}) = S$ describing A.

So we can view S as a system of polynomial equations and inequalities $S(\underline{U}, \underline{T}, \underline{X})$ with coefficient in $\mathbb{Q}$ such that A is the set of solutions in R^{m+n} of the system $S(\underline{u}, \underline{T}, \underline{X})$, i.e.

$$A = \{(\underline{t}, \underline{x}) \in R^{m+n} : (\underline{t}, \underline{x}) \text{ is solution of } S(\underline{u}, \underline{T}, \underline{X})\}.$$

By Tarski-Seidenberg's theorem, we have systems of polynomial equalities and inequalities with coefficients in $\mathbb{Q}$, say

$$S_1(\underline{u}, \underline{T}), \dots, S_l(\underline{u}, \underline{T}),$$

such that for any $\underline{t} \in R^m$ the system $S(\underline{u}, \underline{t}, \underline{X})$ has a solution $\underline{x} = (x_1, \dots, x_n) \in R^n$ if and only if $(\underline{u}, \underline{t})$ is a solution for one of $S_1(\underline{u}, \underline{T}), \dots, S_l(\underline{u}, \underline{T})$, i.e.

$$\begin{aligned}
\pi(A) &= \{\underline{t} \in R^m : \exists\, \underline{x} \in R^n \text{ with } (\underline{t}, \underline{x}) \in A\} \\
&= \{\underline{t} \in R^m : \exists\, \underline{x} \in R^n \text{ s.t. } (\underline{t}, \underline{x}) \text{ is a solution of } S(\underline{u}, \underline{T}, \underline{X})\} \\
&= \{\underline{t} \in R^m : \text{the system } S(\underline{u}, \underline{t}, \underline{X}) \text{ has a solution } \underline{x} \in R^n\} \\
&= \{\underline{t} \in R^m : \underline{t} \text{ is a solution for one of the } S_i(\underline{u}, \underline{T}), i = 1, \dots, l\} \\
&= \bigcup_{i=1,\dots,l} \{\underline{t} \in R^m : \underline{t} \text{ is a solution of } S_i(\underline{u}, \underline{T})\}.
\end{aligned}$$

□

We shall show many important consequences such as the image of a semialgebraic function is semialgebraic and the closure and the interior of a semialgebraic set are semialgebraic.

Definition 2.2. Let $A \subseteq R^m$ and $B \subseteq R^n$. We say that $f\colon A \to B$, is a **semialgebraic map** if A and B are semialgebraic and

$$\Gamma(f) = \{(\underline{x}, \underline{y}) \in R^{m+n} : \underline{x} \in A,\ \underline{y} \in B,\ \underline{y} = f(\underline{x})\}$$

is semialgebraic.

3. Formulas in the language of real closed fields

Definition 3.1. A **first order formula in the language of real closed fields** is obtained as follows recursively:

(1) if $f(\underline{x}) \in \mathbb{Q}[x_1, \dots, x_n]$, $n \geqslant 1$, then

$$f(\underline{x}) \geqslant 0,\ f(\underline{x}) > 0,\ f(\underline{x}) = 0,\ f(\underline{x}) \neq 0$$

are first order formulas (with free variables $\underline{x} = (x_1, \dots, x_n)$);

(2) if Φ and Ψ are first order formulas, then

$$\Phi \wedge \Psi, \quad \Phi \vee \Psi, \quad \neg\Phi$$

are also first order formulas (with free variables given by the union of the free variables of Φ and the free variables of Ψ);

(3) if Φ is a first order formula then

$$\exists x \, \Phi \quad \text{and} \quad \forall x \, \Phi$$

are first order formulas (with the same free variables as Φ minus $\{x\}$).

The formulas obtained using just (1) and (2) are called **quantifier free**.