# POSITIVE POLYNOMIALS LECTURE NOTES (11: 20/05/10 - BEARBEITET 31/01/19)

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## Contents



## 1. ALGEBRAIC INDEPENDENCE AND TRANSCENDENCE DEGREE

**Definition 1.1.** (Recall) Let $E/F$ be a field extension:

(1) $A \subseteq E$ is called **algebraically independent** over $F$ if $\forall\ a_1, \dots, a_n \in A$ there exists no nonzero polynomial $f \in F[X_1, \dots, X_n]$ s.t. $f(a_1, \dots, a_n) = 0$.

(2) $A \subseteq E$ is called a **transcendence basis** of $E/F$ if $A$ is a maximal subset (w.r.t. inclusion) of $E$ which is algebraically independent over $F$.

**Lemma 1.2.** Let $E/F$ be a field extension.

(1) (Steinitz exchange) $S \subseteq E$ is algebraically independent over $F$ iff $\forall\ s \in S$ : $s$ is transcendental over $F(S - \{s\})$ $\big($the subfield of $E$ generated by $S - \{s\}\big)$.

(2) $S \subseteq E$ is a transcendence base for $E/F$ iff $S$ is algebraically independent over $F$ and $E$ is algebraic over $F(S)$. □

*Proof.* Exercise 1 of ÜB 6.

**Theorem 1.3.** The extension $E/F$ has a transcendence base and any two transcendence bases of $E/F$ have the same cardinality.

*Proof.* The existence follows by Zorn's lemma and the second statement uses the Steinitz exchange lemma (above). □

**Definition 1.4.** The cardinality of a transcendence base of $E/F$ is called the **transcendence degree** of $E/F$, denoted by trdeg $(E)$ over $F$.

## 2. KRULL DIMENSION OF A RING

**Definition 2.1** Let $A$ be a commutative ring with 1.

(1) A **chain** of prime ideals of $A$ is of the form
$\{0\} \subseteq \wp_0 \subsetneq \wp_1 \subsetneq \ldots \subsetneq \wp_k \subsetneq \ldots \subsetneq A$, where $\wp_i$ are prime ideals of $A$.

(2) The **Krull dimension** of $A$, denoted by dim $(A)$ is defined to be the maximum $k$ such that there is a chain of prime ideals of length $k$ in $A$, i.e. $\wp_0 \subsetneq \wp_1 \subsetneq \ldots \subsetneq \wp_k$ $\left[\dim(A) \text{ can be infinite if arbitrary long chains}\right]$.

**Theorem 2.2.** Let $F$ be a field and $I$ be any prime ideal in $F[\underline{X}]$. Then

$$\dim\left(\frac{F[\underline{X}]}{I}\right) = \text{trdeg}\left(ff\left(\frac{F[\underline{X}]}{I}\right)\right).$$

□

**Recall 2.3.** For $S \subseteq F^n$

$$\mathcal{I}(S) = \{f \in F[\underline{X}] \mid f(\underline{x}) = 0, \forall\ \underline{x} \in S\}$$

is the ideal of polynomials vanishing on $S$.

**Definition 2.4. Dimension of semi-algebraic sets $\subseteq \mathbb{R}^n$:** Let $K \subseteq \mathbb{R}^n$ be a semi-algebraic set. Then

$$\dim\ (K) := \dim\left(\frac{\mathbb{R}[\underline{X}]}{\mathcal{I}(K)}\right).$$

In the lecture 10 (Proposition 1.2) we have proved the following proposition:

**Proposition 2.5.** Suppose $n \geq 3$. Let $S = \{g_1, \ldots, g_s\}$ be a finite subset of $\mathbb{R}[\underline{X}]$ such that $K_S \subseteq \mathbb{R}^n$ and int$(K_S) \neq \emptyset$. Then there exists $f \in \mathbb{R}[\underline{X}]$ such that $f \geq 0$ on $\mathbb{R}^n$ and $f \notin T_S$ .

This is just a special case of the following result due to Scheiderer:

**Theorem 2.6.** (Scheiderer) (Theorem 1.11 of lecture 10) Let $S$ be a finite subset of $\mathbb{R}[\underline{X}]$ and $K_S \subseteq \mathbb{R}^n$ s.t. dim$K_S \geq 3$. Then there exists $f \in \mathbb{R}[\underline{X}]$; $f \geq 0$ on $\mathbb{R}^n$ and $f \notin T_S$.

To deduce Proposition 2.5 using Theorem 2.6 it suffices to prove the following lemma:

**Lemma 2.7.** Let $K \subseteq \mathbb{R}^n$ be a semi algebraic subset. Then

$$\text{int}(K) \neq \phi \Rightarrow \dim\ (K) = n$$

*Proof.* We have $\dim\ (K) = \dim \left( \dfrac{\mathbb{R}[\underline{X}]}{\mathcal{I}(K)} \right)$, and

we **claim** that $\mathcal{I}(K) = \{0\}$ :

$f \in \mathcal{I}(K) \Rightarrow f = 0$ on $K \Rightarrow f = 0$ on $\underbrace{\text{int}\ (K)}_{(\neq\phi)} \Rightarrow f$ vanishes on a nonempty open set $\Rightarrow f \equiv 0$ (by Remark 2.2 of lecture 2).

So, $\dim\ (K) = \dim\ (\mathbb{R}[\underline{X}]) = \text{trdeg}\ (\mathbb{R}(\underline{X}))$ over $\mathbb{R}$

$= n$ □

## 3. LOW DIMENSIONS

**Proposition 3.1.** Let $n = 2$, $K_S \subseteq \mathbb{R}^2$ and $K_S$ contains a 2-dimensional affine cone. Then $\exists\ f \in \mathbb{R}[X, Y]$; $f \geq 0$ on $\mathbb{R}^2$; $f \notin T_S$.

**Definition 3.2.** (For $n = 1$) Let $K$ be a basic closed semi algebraic subset of $\mathbb{R}$. Then $K$ is a finite union of intervals.
The natural description $S$ of $K$ as basic closed semi algebraic subset is defined as

1. if $a \in \mathbb{R}$ is the smallest element of $K$, then take the polynomial $X - a \in S$
2. if $a \in \mathbb{R}$ is the greatest element of $K$, then take the polynomial $a - X \in S$
3. if $a, b \in K$, $a < b$, $(a, b) \cap K = \phi$, then take the polynomial $(X - a)(X - b) \in S$
4. no other polynomial should be in $S$.

**Proposition 3.3.** Let $K \subseteq \mathbb{R}$ be a basic closed semi algebraic subset and $S$ is the natural description of $K$. Then $\forall\ f \in \mathbb{R}[X]$ :

$$f \geq 0 \text{ on } K \Rightarrow f \in T_S,$$

i.e. for every basic semi algebraic subset $K$ of $\mathbb{R}$, there exists a description $S$ (namely the natural) so that $T_S$ is saturated.

**Proposition 3.4.** Let $K \subseteq \mathbb{R}$ be a non-compact basic semi algebraic subset and $S^{'}$ be a description of $K$. Then

$$T_{S'} \text{ is saturated} \Leftrightarrow S^{'} \supseteq S \text{ (up to a scalar multiple factor).}$$

**Remark 3.5.** Summarizing:

(1) $\dim(K_S) \geq 3 \Rightarrow T_S$ is not saturated.

(2) $\dim(K_S) = 2 \Rightarrow T_S$ can be or cannot be saturated (depending on the geometry of $K$ and $S$).

(3) $\dim(K_S) = 1 \Rightarrow T_S$ can be or cannot be saturated [but depends on $K$ and description $S$ of $K$).

After all this discussion about positive polynomials, strictly positive polynomials, we now want to show **Schmüdgen's Positivstellensatz**:

**Theorem 3.6.** (Schmüdgen's Positivstellensatz) Let $S = \{g_1, \dots, g_s\}$ be a finite subset of $\mathbb{R}[X_1, \dots, X_n]$ and $K_S \subseteq \mathbb{R}^n$ be a compact basic closed semi algebraic set. And let $f \in \mathbb{R}[\underline{X}]$ s.t. $f > 0$ on $K_S$. Then $f \in T_S$.

**Note** that this holds for every finite description $S$ of $K$.

To prove this we first need Representation Theorem (Stone-Krivine, Kadison-Dubois), which will be proved in the next lecture.