CYCLIC 2-STRUCTURES AND SPACES OF ORDERINGS OF POWER SERIES FIELDS IN TWO VARIABLES

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Abstract. We consider the space of orderings of the field $R((x,y))$ and the space of orderings of the field $R((x))(y)$, where R is a real closed field. We examine the structure of these objects and their relationship to each other. We define a cyclic 2-structure to be a pair (S,Φ) where S is a cyclically ordered set and Φ is an equivalence relation on S such that each equivalence class has exactly two elements. We show that each of these spaces of orderings is described by a cyclic 2-structure, in a natural way. We also show that if the real closed field R is archimedean then the space of $\mathbb{R}$-places of these fields is describable in terms of the cyclic 2-structure.

1. Introduction

For a formally real field K, $\operatorname{Sper} K$ denotes the set of orderings of K, M_K denotes the set of $\mathbb{R}$-places of K, and $\lambda : \operatorname{Sper} K \to M_K$ denotes the natural map. See [3] [15] [16] or [20] for a more precise description of these objects and for basic terminology and basic results. $\dot{K}$ denotes the multiplicative group $K\backslash\{0\}$. $\operatorname{Sper} K$ and M_K are topological spaces. $\operatorname{Sper} K$ is a boolean space. The harrison sets

$$H_K(f) := \{P \in \operatorname{Sper} K \mid f \in P\},\ f \in \dot{K},$$

form a subbasis for the topology on $\operatorname{Sper} K$. M_K is compact and hausdorff. λ is continuous and surjective. The topology on M_K is the quotient topology.

For what we do here, knowledge of abstract spaces of orderings [2] [16] is optional. All we need is the definition of the space of orderings of a formally real field. For $f \in \dot{K}$, define $\overline{f} : \operatorname{Sper} K \to \{-1,1\}$ by

$$\overline{f}(P) := \begin{cases} 1 \text{ if } f \in P, \\ -1 \text{ if } f \in -P \end{cases} .$$

The topology on $\operatorname{Sper} K$ is the weakest topology making the functions $\overline{f}$ continuous, giving $\{-1,1\}$ the discrete topology. The *space of orderings* of K is the pair $(\operatorname{Sper} K, G_K)$, where G_K is the group of all functions $\overline{f}$, $f \in \dot{K}$.

Orderings and real places arise most naturally in the context of real algebraic geometry [2] [4] [5] [13] [17] [20]. Let R be a real closed field, e.g., take $R = \mathbb{R}$. The formal power series ring $R[[x_1,\dots,x_d]]$ also arises naturally in this context, as the completion of the coordinate ring of a d-dimensional algebraic variety over R at a non-singular point. $R((x_1,\dots,x_d))$ denotes the field of fractions of the integral domain $R[[x_1,\dots,x_d]]$.

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We restrict our attention here to the case $d = 2$. Orderings on $\mathbb{R}((x,y))$ and on $\mathbb{R}((x,y))_{\text{an}}$, the field of fractions of the ring $\mathbb{R}[[x,y]]_{\text{an}}$ of convergent power series, are considered already in [1]. More recently, in [8], orderings on $\mathbb{R}((x,y))$ are exploited to prove a representation result for polynomials non-negative on a compact basic semialgebraic subset of $\mathbb{R}^2$, extending an earlier such result in [22].

Our main results are Theorems 5.1 and 6.5. The study of orderings and $\mathbb{R}$-places on $R((x,y))$ reduces by an application of the Weierstrass Preparation Theorem, see Theorem 2.1, to the study of orderings and $\mathbb{R}$-places on $R((x))(y)$. It is a consequence of this that the structure of the space of orderings and of the space of $\mathbb{R}$-places of these two fields are closely interrelated. We introduce the idea of a cyclic 2-structure in Section 5 and show, in Theorem 5.1, how each of these spaces of orderings is described by a cyclic 2-structure, in a natural way. In Section 6, which is the most technically demanding section in the paper, we apply ideas from [14] to understand the fibers of the map λ in this situation. We explain, in Theorem 6.5, how the space of $\mathbb{R}$-places is describable in terms of the cyclic 2-structure if R is archimedean. This is an interesting result, more especially so in view of the well-known fact that the space of $\mathbb{R}$-places is typically *not* describable in terms of the space of orderings. We give an example, see Example 6.6, showing how Theorem 6.5 fails if R is not archimedean.

Denote by $R((x,y))_{\text{alg}}$ the field of fractions of the ring $R[[x,y]]_{\text{alg}}$ of algebraic power series [5, Ch. 8]. We do not consider $\mathbb{R}((x,y))_{\text{an}}$ or $R((x,y))_{\text{alg}}$ explicitly in what we do here. But it still needs to be mentioned that everything we do here for $R((x,y))$ carries over with suitable modifications to these fields.

In [11] and [12] it is asked if the pp conjecture holds for the space of orderings of $R((x,y))$. We do not consider this question, although the results we do obtain might provide the basis for an eventual answer to this question.

2. Preparation theorem and factorization

Throughout the paper R denotes a real closed field. The results in Section 2 are well-known and are valid for any field R.

A monic polynomial $f \in R[[x]][y]$ of the form

$$f = y^n + \sum_{i=0}^{n-1} a_i(x)y^i,\ a_i(x) \in R[[x]],\ x \mid a_i(x),\ 0 \le i < n,\ n \ge 0$$

will be called *distinguished*.

Theorem 2.1. [Preparation Theorem] *Every non-zero element $f \in R[[x,y]]$ has a unique decomposition*

$$f = ux^k f^\star,$$

where u is a unit in $R[[x,y]]$, $k \ge 0$ and $f^\star$ is a distinguished polynomial in $R[[x]][y]$.

See [23, Cor 1, p. 145] for the proof. See [23, Cor. 1, p. 131] for a description of the units.

Remark 2.2. The field $R((x))$ is a complete discrete valued field with residue field R. Let $R((x))^{\text{ac}}$ denote the algebraic closure of $R((x))$ and let v denote the unique extension of the valuation to $R((x))^{\text{ac}}$.

(1) Let $f \in R[[x]][y]$ be distinguished, $f = y^n + \sum_{i=0}^{n-1} a_i(x)y^i$. If $r \in R((x))^{\text{ac}}$ and $v(r) \le 0$ then $v(r^n) < v(a_i y^i)$, $i = 1, \dots, n-1$, so $v(f(r)) = v(y^n) \le 0$. In particular, all roots of f have positive value.

(2) Conversely, if $f \in R((x))[y]$ is monic and all the roots of f have positive value then f is distinguished (because the coefficients $a_1, \dots, a_{n-1}$ of f are elementary symmetric functions of the roots, so they also have positive value).

(3) In particular, if $f \in R((x))[y]$ is monic and irreducible and one root of f has positive value then all roots of f have positive value (because the various roots are conjugate to each other, so they have the same value) so f is distinguished.

Lemma 2.3. *If $f \in R[[x]][y]$ is distinguished, then the following conditions are equivalent:*

(1) *f is irreducible in $R[[x,y]]$,*
(2) *f is irreducible in $R[[x]][y]$,*
(3) *f is irreducible in $R((x))[y]$.*

Proof. Since $R[[x]]$ is a UFD and f has content 1 (because it is monic), (2) $\Leftrightarrow$ (3) is clear. (1) $\Rightarrow$ (2): Suppose f is irreducible in $R[[x,y]]$ and $f = gh$, $g, h \in R[[x]][y]$. Scaling by a unit of $R[[x]]$ we may assume g and h are monic so, by Remark 2.2, parts (1) and (2), g and h are distinguished. One of g, h is a unit in $R[[x,y]]$, say g is a unit in $R[[x,y]]$. Since g is also distinguished, this forces $g = 1$, i.e., g is already a unit in $R[[x]][y]$. (2) $\Rightarrow$ (1): By [23, Cor. 2, p. 146], the ring homomorphism $R[[x]][y] \to R[[x,y]]/(f)$ induced by the inclusion $R[[x]][y] \subseteq R[[x,y]]$ is surjective and has kernel equal to the principal ideal in $R[[x]][y]$ generated by f (which, by abuse of notation, we also denote by (f)), so $R[[x,y]]/(f) \cong R[[x]][y]/(f)$. We know that $R[[x]][y]$ is a UFD. If f is irreducible in $R[[x]][y]$ then the principal ideal in $R[[x]][y]$ generated by f is prime, so the principal ideal in $R[[x,y]]$ generated by f is also prime. This implies that f is irreducible in $R[[x,y]]$. □

The ring $R[[x,y]]$ is a UFD [23, Th. 6, p. 148]. This can be deduced from the fact that $R[[x]][y]$ is a UFD, by combining Theorem 2.1 and Lemma 2.3. Each non-zero $f \in R[[x,y]]$ factors uniquely as

$$f = ux^k f_1 \cdots f_m$$

where u is a unit of $R[[x,y]]$, $k \geq 0$, $m \geq 0$ and each $f_j \in R[[x]][y]$ is distinguished and irreducible.

We record the following consequence of the proof of Lemma 2.3. See also [23, Cor., p. 149].

Corollary 2.4. *If $f \in R[[x]][y]$ is distinguished and irreducible, then the field of fractions of $R[[x,y]]/(f)$ is canonically isomorphic to $R((x))[y]/(f)$.*

3. THE CONJUGATION MAP

The field $R((x))$ has two orderings, one making $x > 0$, and one making $x < 0$. Denote the associated real closures by R_1 and R_2, respectively. Any finite extension L of $R((x))$ is a complete discrete valued field with residue field R or C, where $C := R(\sqrt{-1})$. If the residue field is R then L has two orderings, by the Baer-Krull Theorem [16, Sect. 1.3] [17, Sect. 1.5] . If the residue field is C then $\sqrt{-1} \in L$ and L has no orderings. Suppose now that $L = R((x))[y]/(f)$, where $f \in R((x))[y]$ is irreducible. Suppose L is formally real, i.e., the prime ideal (f) is real. Orderings of L correspond to roots of f in $R_1 \dot\cup R_2$ (disjoint union). Either there are two roots of f in R_1 and none in R_2 or two in R_2 and none in R_1 or one in R_1 and one in R_2.

Putting it another way, if $r \in R_1 \dot\cup R_2$ and f denotes the minimal polynomial of r over $R((x))$, then f has another root $r' \in R_1 \dot\cup R_2$. In this way we have a well-defined map $r \mapsto r'$ from $R_1 \dot\cup R_2$ onto itself, which we call the *conjugation map*.

By Puiseux's Theorem, each $r \in R_1$ (resp., $r \in R_2$) is expressible as

$$r = \sum_{i=k}^{\infty} a_i x^{i/d} \text{ (resp., } r = \sum_{i=k}^{\infty} a_i (-x)^{i/d}),$$

$a_i \in R$, $d :=$ the degree of the minimal polynomial of r over $R((x))$. The integer d is also described as the least common denominator of the fractions i/d with $a_i \neq 0$.

By Kummer Theory, for $r = \sum a_i x^{i/d}$, as above, the conjugates of r over $R((x))$ (or equivalently, over $C((x))$) have the form $\sum a_i \omega^i x^{i/d}$ where ω is a d-th root of 1. If d is even, -1 is a d-th root of 1, and $r' = \sum a_i (-1)^i x^{i/d}$. If d is odd then $\mu := -\frac{(-x)^{1/d}}{x^{1/d}}$ is a d-th root of 1, and $r' = \sum a_i \mu^i x^{i/d} = \sum a_i (-1)^i (-x)^{i/d}$. Similar formulas hold for $r = \sum a_i (-x)^{i/d}$.

In summary, the map $r \mapsto r'$ from $R_1 \dot\cup R_2$ to $R_1 \dot\cup R_2$ is given by

$$\sum a_i x^{i/d} \mapsto \sum a_i (-1)^i x^{i/d}, \quad \sum a_i (-x)^{i/d} \mapsto \sum a_i (-1)^i (-x)^{i/d}$$

if d is even and

$$\sum a_i x^{i/d} \mapsto \sum a_i (-1)^i (-x)^{i/d}, \quad \sum a_i (-x)^{i/d} \mapsto \sum a_i (-1)^i x^{i/d}$$

if d is odd. If $d = 1$ then $r \in R((x))$ so there is one copy of r in R_1 and one in R_2 and, in this case, the map $r \mapsto r'$ just interchanges these two copies.

Remark 3.1. If $R = \mathbb{R}$, the irreducible polynomial $f \in R((x))[y]$ is distinguished and the coefficients of f are analytic functions of x in a neighborhood of 0 then $y = r$ and $y = r'$ (where r, r' are the real conjugate roots of f) are precisely the real half-branches of the plane curve $f(x, y) = 0$ at $(0, 0)$. The same is true for $R \neq \mathbb{R}$, if the irreducible polynomial f is distinguished and the coefficients of f are Nash functions of x in a neighborhood of 0.

Theorem 3.2. [Continuity of Conjugation] *For each $r \in R_1 \dot\cup R_2$ and each neighborhood U of $\{r, r'\}$ in $R_1 \dot\cup R_2$, there is a neighborhood V of $\{r, r'\}$ in $R_1 \dot\cup R_2$ contained in U and invariant under conjugation.*

Here, the topology on $R_1 \dot\cup R_2$ is the disjoint union topology, giving each R_i the order topology.

Proof. Since r belongs to R_1 or R_2 and, similarly, r' belongs to R_1 or R_2, there are four cases to consider. We consider the case $r \in R_1$, $r' \in R_1$. The other cases are similar. Thus $r = \sum a_i x^{i/d}$, $r' = \sum a_i (-1)^i x^{i/d}$. Choose $V = V_1 \cup V_2$ where $V_1 := \{s \in R_1 \mid v(s - r) > \gamma\}$ and $V_2 := \{s \in R_1 \mid v(s - r') > \gamma\}$, γ large enough so that $V \subseteq U$ and d is the least common denominator of the fractions $\{i/d \mid i/d < \gamma,\ a_i \neq 0\}$. The point is that if $s \in V$ then s coincides with either r or r' up to terms of value $\geq \gamma$ and the degree of s is some multiple of d. If the degree of s is an even multiple of d then s' is in the same part of V as s. If the degree of s is an odd multiple of d then s' is in the other part of V. □

Remark 3.3. Consider the intervals V_i^-, V_i^+, $i = 1, 2$ defined by $V_1^- = \{s \in V_1 \mid s < r\}$, $V_1^+ = \{s \in V_1 \mid s > r\}$, $V_2^- = \{s \in V_2 \mid s < r'\}$, $V_2^+ = \{s \in V_2 \mid s > r'\}$. For each pair V_i^ϵ, V_j^δ, $i, j \in \{1, 2\}$, $\epsilon, \delta \in \{+, -\}$, there are elements of V_i^ϵ which are mapped to V_j^δ by conjugation.

4. ORDERINGS

Let $(S,<)$ be an ordered set. A *cut* of $(S,<)$ is a pair (A,B) where A,B are subsets of S, $A\cup B=S$, and $A<B$. A cut said to be *proper* if A and B are both non-empty. The two *principal cuts* determined by an element $r\in S$ are

$$r_- := (\{a \mid a<r\},\{b\mid b\geq r\}) \text{ and } r_+ := (\{a\mid a\leq r\},\{b\mid b>r\}).$$

The set of cuts of an ordered set $S=(S,<)$ will be denoted by $C(S)$. The following result appears to be well-known.

Lemma 4.1. *For any ordered set S, the set of cuts of S equipped with its natural order topology is a boolean space.*

Proof. Define $\Psi : C(S)\to\{0,1\}^S$ by

$$\Psi(A,B)(r)=\begin{cases}0 \text{ if } r\in A\\ 1 \text{ if } r\in B\end{cases}.$$

One checks that Ψ is injective and that the topology on $C(S)$ is induced by Ψ and the product topology on $\{0,1\}^S$, giving $\{0,1\}$ the discrete topology. It follows that $C(S)$ is totally disconnected. In view of Tychonoff's Theorem, to show $C(S)$ is compact it suffices to show the image of $C(S)$ under Ψ is closed in $\{0,1\}^S$. This is straightforward to check. □

Remark 4.2. For a formally real field K, the set $\operatorname{Sper} K(y)$ is naturally identified with the disjoint union of the sets $\operatorname{Sper} R(y)$, where R runs through the set of real closures of K [9, Lemma 8]. The natural bijection $\dot\cup_R \operatorname{Sper} R(y)\to \operatorname{Sper} K(y)$ is continuous, where $\dot\cup_R \operatorname{Sper} R(y)$ is given the topology of the disjoint union. If $\operatorname{Sper} K$ is finite then the disjoint union is compact, and the bijection is a homeomorphism. The orderings of $R(y)$ are naturally identified with the cuts of R [9] [10]. The topology on $C(R)$ induced by the harrison topology on $\operatorname{Sper} R(y)$ coincides with the order topology on $C(R)$.

Let R_1, R_2 be the two real closures of $R((x))$ as defined in the previous section. Consider the topological space of orderings of the field $R((x))(y)$. By Remark 4.2 we have

$$\operatorname{Sper} R((x))(y)=\operatorname{Sper} R_1(y)\,\dot\cup\, \operatorname{Sper} R_2(y)=C(R_1)\,\dot\cup\, C(R_2).$$

Set $I_j := \{r\in R_j \mid v(r)>0\}$, $j=1,2$. Here, v denotes the extension to R_j of the standard discrete valuation on $R((x))$, i.e., I_j is the set of elements of R_j which are infinitely small relative to elements of R.

We will prove that $\operatorname{Sper} R((x,y))$ is identified with $C(I_1)\dot\cup C(I_2)$. We begin by proving some preliminary results. Viewing $R((x))(y)$ as a subfield of $R((x,y))$, we have the natural continuous restriction map $\rho : \operatorname{Sper} R((x,y))\to \operatorname{Sper} R((x))(y)$.

Lemma 4.3. *The map ρ is injective.*

Proof. Suppose that P_1, P_2 are two different orderings of $R((x,y))$. There exists $f\in R[[x,y]]$ which separates P_1 and P_2. By the Preparation Theorem $f=ux^kf^\star$, where $f^\star$ is a distinguished polynomial of $R[[x]][y]$, $k\geq 0$, and u is a unit of $R[[x,y]]$. u has the form $u=a+w$, $a\in R$, $a\neq 0$, w an element of the maximal ideal of $R[[x,y]]$. If $a>0$ then u is a square, and conversely [17, Prop. 1.6.2]. It follows that u is $\pm$ a square so the sign of u is the same at P_1 and P_2. Consequently, the element $x^kf^\star\in R[[x]][y]$ is also a separating element for P_1 and P_2. □

A unit of $R[[x,y]]$ having the form $u = a + w$, $a \in R$, $a > 0$, w an element of the maximal ideal of $R[[x,y]]$, will be referred to as a *positive unit* of $R[[x,y]]$.

Lemma 4.4. *The image of* $\operatorname{Sper} R((x,y))$ *under* ρ *is a subset of* $C(I_1)\dot{\cup}C(I_2)$

Proof. Let P be an ordering of $R((x,y))$. The restriction of P to $R((x))(y)$ extends to $R_j(y)$ for $j = 1$ or 2. Denote this extension by Q. Fix a positive element $r \in R_j$, $v(r) \leq 0$. r is bounded below by a positive element a of R. (If $j = 1$, resp., $j = 2$, write $r = bx^{k/d}+$ terms of higher value, resp., $r = b(-x)^{k/d}+$ terms of higher value, where $b \in R$, $b \neq 0$. Take $a = b/2$.) $a \pm y$ is a unit and a square in $R[[x,y]]$ so $a \pm y \in P$. It follows that $r \pm y = (r - a) + (a \pm y) \in Q$. Since this is valid for any positive $r \in R_j$ with $v(r) \leq 0$, it follows that the cut of R_j determined by Q is actually a cut of I_j. □

Theorem 4.5. *The map* $\rho : \operatorname{Sper} R((x,y)) \to C(I_1)\dot{\cup}C(I_2)$ *is a homeomorphism.*

Proof. In view of Lemmas 4.3 and 4.4 it remains to show that each element of $C(I_1)\dot{\cup}C(I_2)$ is in the image of ρ. We begin by considering the case of a principal cut in I_1 determined by $r \in I_1$. The general case will follow from this by a compactness argument. Let f be the minimal polynomial of r over $R((x))$. By Remark 2.2 (3), f is distinguished. By Lemma 2.3, f is irreducible in $R[[x,y]]$. Since $R[[x,y]]$ is UFD, the localization $R[[x,y]]_{(f)}$ is a discrete valuation ring of $R((x,y))$ with residue field equal to the field of fractions of $R[[x,y]]/(f)$ which, by Corollary 2.4, is canonically identified with $R((x))[y]/(f)$. The latter field is a complete discrete valued field with exactly 2 orderings. The ordering we are interested in is the ordering, call it P, on $R((x))[y]/(f)$ induced by the embedding of $R((x))[y]/(f)$ into R_1 defined by $y+(f) \mapsto r$. By the Baer-Krull Theorem, there are exactly 2 orderings of $R((x,y))$ compatible with the discrete valuation ring $R[[x,y]]_{(f)}$ and pushing down to the ordering P. The two orderings of $R((x))(y)$ obtained from these two orderings by restriction are precisely the two orderings of $R((x))(y)$ compatible with the discrete valuation ring $R((x))[y]_{(f)}$ and pushing down to the ordering P on the residue field $R((x))[y]/(f)$. These, in turn, are precisely the two orderings coming from the two principal cuts of R_1 corresponding to r.

Let $i_1 : \operatorname{Sper} R_1(y) \hookrightarrow \operatorname{Sper} R((x))(y)$ be the canonical restriction. For any non-principal proper cut (A,B) of I_1 consider the family of sets

$$H(r_1,r_2) = \rho^{-1}(i_1(H_{R_1(y)}(y - r_1) \cap H_{R_1(y)}(r_2 - y))),$$

where $H_{R_1(y)}(y-r_1)$ and $H_{R_1(y)}(r_2-y)$ are harrison subbasis sets of the topological space $\operatorname{Sper} R_1(y)$, $r_1 \in A$, $r_2 \in B$. Since the maps ρ and i_1 are continuous, the sets $H(r_1,r_2)$ are closed, and they are non-empty because $H(r_1,r_2)$ contains the inverse image of the orderings of $R((x))(y)$ determined by the principal cuts associated to r, for every $r_1 < r < r_2$. Note that if $r_1, s_1 \in A$ and $r_2, s_2 \in B$ then $H(r_1,r_2) \cap H(s_1,s_2) = H(\max\{r_1,s_1\},\min\{r_2,s_2\})$. Thus the family is closed under finite intersections. By compactness of the space of orderings this family has a non-empty intersection.

For improper cuts, consider the families:

$$H(r_1) = \rho^{-1}(i_1(H_{R_1(y)}(y - r_1))),\ \ r_1 \in I_1$$

and

$$H(r_2) = \rho^{-1}(i_1(H_{R_1(y)}(r_2 - y))),\ \ r_2 \in I_1.$$

Each of these families is a nested family of non-empty closed sets. By compactness, the intersection of each of these families is non-empty.

This shows that the image of ρ contains $C(I_1)$. A similar argument shows that the image of ρ contains $C(I_2)$. $\square$

Here is a less cluttered description of the image of ρ:

Corollary 4.6. *The image of* $\operatorname{Sper} R((x,y))$ *under* ρ *is equal to the set of orderings* P *of* $R((x))(y)$ *satisfying* $a \pm y \in P$ *for all positive* $a \in R$.

Proof. Suppose P is an ordering of $R((x))(y)$ satisfying $a \pm y \in P$ for all positive $a \in R$. P extends to an ordering Q of $R_j(y)$ for $j = 1$ or 2. The argument in the proof of Lemma 4.4 shows that the cut of R_j determined by Q is actually a cut of I_j. Theorem 4.5 then implies P is in the image of ρ. The other inclusion is immediate from the fact that for any positive $a \in R$, $a \pm y$ is a positive unit in $R[[x,y]]$, so it is a square. $\square$

Remark 4.7. Using the Preparation Theorem together with the fact that every unit of $R[[x,y]]$ is $\pm$ a square, we see that the homomorphism $G_{R((x))(y)} \to G_{R((x,y))}$ induced by the inclusion $R((x))(y) \subseteq R((x,y))$ is surjective. Combining this with Corollary 4.6, we see that $(\operatorname{Sper} R((x,y)), G_{R((x,y))})$ is identified via ρ with the subspace $(Y, G_{R((x))(y)}|_Y)$ of $(\operatorname{Sper} R((x))(y), G_{R((x))(y)})$, where

$$Y := \cap_{a\in R, a>0}(H_{R((x))(y)}(a+y) \cap H_{R((x))(y)}(a-y)).$$

See [16, p. 32-33] for basic material on subspaces.

5. Cyclic 2-structures

By a *cyclically ordered set* we mean a set S equipped with a ternary relation such that

(1) $\forall\ a,b,c \in S\ a<b<c \Rightarrow a \neq b \neq c \neq a$.
(2) $\forall\ a,b,c \in S\ a<b<c \Rightarrow b<c<a$.
(3) $\forall\ c \in S$, the set $S\backslash\{c\}$ is totally ordered via $a<b$ iff $a<b<c$.[1]

For a cyclically ordered set S and $a,b \in S$, $a \neq b$, the *interval* (a,b) in S is defined to be the totally ordered set $\{x \in S \mid a<x<b\}$. Cuts of S are defined to be cuts of intervals in S identified in the obvious way. The set of all cuts of a cyclically ordered set S, denoted $C(S)$, is itself a cyclically ordered set. It is a boolean space which is identified naturally with the boolean space consisting of all cuts of the totally ordered set $S\backslash\{c\}$ for any $c \in S$; see Lemma 4.1.

By a *cyclic 2-structure* we mean a pair (S,Φ) consisting of a cyclically ordered set S together with an equivalence relation Φ on S such that each equivalence class has exactly two elements. A priori no connection between the equivalence relation and the ordering is assumed. For $r \in S$, denote by r' the other element of the equivalence class of r. We refer to r' as the *conjugate* of r. The mapping from S to S defined by $r \mapsto r'$ will be called *the conjugation map.* It is idempotent with no fixed points. Each equivalence class $\{r,r'\}$ determines two arcs $(r,r') = \{s \in S \mid r<s<r'\}$ and $(r',r) = \{s \in S \mid r'<s<r\}$ and two functions $f_1, f_2 : C(S) \to \{-1,1\}$ (called the *atoms* associated to equivalence class $\{r,r'\}$) defined by

$$f_1(x) := \begin{cases} 1 \text{ if } x \text{ is a cut of } (r,r'), \\ -1 \text{ if } x \text{ is a cut of } (r',r) \end{cases}$$

[1]The idea of a cyclically ordered set is obviously not new. See [19] and [21].

and $f_2 := -f_1$. Note: The principal cuts r_+ and r'_- are to be viewed as cuts of (r, r'). Similarly, the principal cuts r_- and r'_+ are to be viewed as cuts of (r', r). Denote by $G_{(S,\Phi)}$ the group of functions $f : C(S) \to \{-1, 1\}$ generated by the constant functions $1, -1$ and the various atoms determined from the various equivalence classes of S. A pair (X, G), where X is a set and G is a group of functions from X to $\{-1, 1\}$, is said to be *described by the cyclic 2-structure* (S, Φ) if there exists a bijection $p : X \to C(S)$ such that $G = \{f \circ p \mid f \in G_{(S,\Phi)}\}$.

Theorem 5.1. *For any real closed field R, the spaces of orderings of the fields $R((x))(y)$ and $R((x, y))$ are described by cyclic 2-structures in a natural way.*

Proof. We first give the proof for $R((x))(y)$. Let R_1, R_2 be the two real closures of $R((x))$ defined as in Section 3. Define S to be $R_1 \dot\cup R_2 \dot\cup \{-\infty, \infty\}$ (disjoint union) where $-\infty$ and ∞ are new symbols, and order S cyclically so that $\infty < R_1 < -\infty < R_2 < \infty$. Here, the ordering on R_1 is taken to be the opposite of the usual one and the ordering on R_2 is taken to be the usual one. $C(S)$ is identified with $C(R_1) \dot\cup C(R_2)$ which, as was explained in Section 4, is identified with $\operatorname{Sper} R((x))(y)$. Set up the equivalence relation on S so that ∞ and $-\infty$ are in the same class (note that $\pm\overline{x}$ are the two associated atoms) and, for $r \in S$, $r \neq \pm\infty$, r' = the conjugate of r described in Section 3 (recall that r and r' have the same minimal polynomial f over $R((x))$, and note that $\pm\overline{f}$ are the two associated atoms). Any non-zero $u \in R((x))$ is, up to a square, either ± 1 or $\pm x$. An irreducible f of $R((x))[y]$ is, after scaling by a suitable non-zero element of $R((x))$, a monic irreducible. f is either real or non-real. If f is real it is the minimal polynomial over $R((x))$ of some unique pair $\{r, r'\}$ as above. If f is non-real then f is a sum of two squares in $R((x))[y]$ (see [17, p. 19]), so $\overline{f}$ does not contribute to $G_{R((x))(y)}$ in this case.

The proof for $R((x, y))$ is similar. We take $S = I_1 \dot\cup I_2 \dot\cup \{-\infty, \infty\}$ (disjoint union), where $I_i \subseteq R_i$ is the set of infinitesimal elements of R_i, $i = 1, 2$, notation as in Section 4. We order S cyclically so that $\infty < I_1 < -\infty < I_2 < \infty$. Here, the ordering on I_1 is taken to be the opposite of the usual one and the ordering on I_2 is taken to be the usual one. $C(S)$ is identified with $C(I_1) \dot\cup C(I_2)$ which, by Theorem 4.5, is identified with $\operatorname{Sper} R((x, y))$. Set up the equivalence relation on S as in the previous paragraph. For any unit u of $R[[x, y]]$, $\overline{u}$ is one of the constant functions ± 1. An irreducible f of $R[[x, y]]$ is (up to a unit) either x or a distinguished irreducible. In the latter case, f is real or non-real. If f is real it is the minimal polynomial over $R((x))$ of some unique pair $\{r, r'\}$ as above. If f is non-real then f is a sum of two squares in $R[[x]][y]$, so $\overline{f}$ does not contribute to $G_{R((x,y))}$. □

Remark 5.2. The cyclic 2-structures (S, Φ) defined in the proof of Theorem 5.1 satisfy various additional properties. There are the constraints provided by Theorem 3.2 and Remark 3.3. There are variants of Theorem 3.2 and Remark 3.3 which hold with $\{r, r'\}$ replaced by $\{\infty, -\infty\}$. There are also constraints coming from the fact that $(C(S), G_{(S,\Phi)})$ is the space of orderings of a field, so it is a space of orderings, i.e., it satisfies axioms AX1, AX2 and AX3 (see [16, p. 21-22]) or, equivalently, axioms (α), (β) and (γ) (see [16, p. 26]).

6. Orderings and $\mathbb{R}$-places

Let K be a formally real field, $\operatorname{Sper} K$ the topological space of orderings of K, M_K the space of $\mathbb{R}$-places of K, $\lambda : \operatorname{Sper} K \to M_K$ the natural map. Recall that λ

is continuous and surjective [15] [16] [20]. A subset Y of $\operatorname{Sper} K$ is called a *fan* if $Y \neq \emptyset$ and every character χ of the group $\dot{K}/\bigcap\{\dot{P} \mid P \in Y\}$ such that $\chi(-1) = -1$ is a signature of some ordering $P \in Y$. Here, $\dot{P} := P\backslash\{0\}$. A fan $Y \subseteq \operatorname{Sper} K$ is said to be *trivial* if contains at most 2 orderings. The *stability index* $s(K)$ of K is defined as the maximum $n \in \mathbb{N}$ such that there exists a fan $Y \subseteq \operatorname{Sper} K$ which contains 2^n orderings (or ∞ if no such finite n exists). There are various equivalent definitions of the stability index; see [6] and [7] or [2] or [15] or [16].

Interest in the stability index derives, in no small part, from its application to minimal generation of semialgebraic sets and semianalytic sets. This is explained in detail in [2]. The following result is well-known.

Theorem 6.1.
(1) The stability index of $R((x))(y)$ is equal to 2.
(2) The stability index of $R((x, y))$ is equal to 2.

Proof. Any finite extension L of $R((x))$ which is formally real has two orderings, so has stability index 1. It follows from this using [6, Satz 4.6] (see also [2, Th. 2.7, Ch. 6]) that the stability index of $R((x))(y)$ is at most 2. (Note: There is a misprint in the statement of [6, Satz 4.6]; $s(K)$ should be $s(F)$.) There are lots of 4-element fans in $\operatorname{Sper} R((x, y))$, e.g., if $f \in R[[x, y]]$ is an irreducible which is distinguished and real, the orderings of $R((x, y))$ compatible with the DVR $R[[x, y]]_{(f)}$ form a 4-element fan.

Claim: For any fan Y in $\operatorname{Sper} R((x, y))$, the image Y' of Y under the natural embedding $\operatorname{Sper} R((x, y)) \hookrightarrow \operatorname{Sper} R((x))(y)$ is a fan in $\operatorname{Sper} R((x))(y)$. Consider the group homomorphism

$$\iota : R((\dot{x}))(y)/\cap\{\dot{P}' \mid P' \in Y'\} \to R((\dot{x}, y))/\cap\{\dot{P} \mid P \in Y\}$$

induced by the inclusion $R((x))(y) \subseteq R((x, y))$. Exploiting the Preparation Theorem and the fact that each unit of $R[[x, y]]$ is $\pm$ a square, we see that ι is surjective. ι is clearly injective. Using these facts together with the fact that Y is a fan we see that Y' is also a fan. This proves the claim.

Putting all these things together yields $2 \leq s(R((x, y))) \leq s(R((x))(y))) \leq 2$, so $s(R((x, y))) = s(R((x))(y))) = 2$. □

By the Baer-Krull Theorem, for each $\xi \in M_K$, the fiber $\lambda^{-1}(\xi)$ is a fan, and the elements of $\lambda^{-1}(\xi)$ are in one-to-one correspondence with characters of the group $V/2V$, where V denotes the value group of the valuation associated to λ. If the stability index of K is equal to n, then every fiber $\lambda^{-1}(\xi)$ contains at most 2^n elements.

Corollary 6.2. *For K equal to $R((x))(y)$ or $R((x, y))$, the fibers $\lambda^{-1}(\xi)$ of the map $\lambda : \operatorname{Sper} K \to M_K$ have at most 4 elements.*

It follows from Corollary 6.2 that the mapping λ is either 1-1, 2-1, or 4-1. At which points is it 1-1? At which points is it 2-1? At which points is it 4-1? We work now to develop a refined version of Corollary 6.2, see Theorem 6.4 below, which answers these questions.

To understand the map $\lambda : \operatorname{Sper} R((x, y)) \to M_{R((x,y))}$, it suffices to understand the map $\lambda : \operatorname{Sper} R((x))(y) \to M_{R((x))(y)}$. We explain this now.

Lemma 6.3. *For any ordering P of $R((x,y))$, the value group of the valuation of $R((x,y))$ associated to P coincides with the value group of the valuation of $R((x))(y)$ associated to the restriction of P to $R((x))(y)$.*

Proof. Any positive unit of $R[[x,y]]$ has the form $a+w$ where a is a positive element of R and w is an element of the maximal ideal of $R[[x,y]]$. For any $n \in \mathbb{N}$, $\frac{1}{n} \pm \frac{w}{a}$ is a unit and a square in $R[[x,y]]$, so $\frac{1}{n} \pm \frac{w}{a} \in P$, i.e., $v_P(\frac{w}{a}) > 0$, i.e., $v_P(a+w) = v_P(a)$, where v_P denotes the valuation of $R((x,y))$ associated to P. The result follows from this, using Theorem 2.1. □

Consider now the commutative diagram

$$\begin{array}{ccc} \text{Sper}\, R((x,y)) & \longrightarrow & \text{Sper}\, R((x))(y) \\ \downarrow \lambda & & \downarrow \lambda \\ M_{R((x,y))} & \longrightarrow & M_{R((x))(y)}, \end{array}$$

the horizontal maps coming from the inclusion $R((x))(y) \subseteq R((x,y))$. By Lemma 4.3 the upper horizontal map in injective. Coupling this with Lemma 6.3 and the Baer-Krull Theorem, we see that the lower horizontal map is also injective and, for each $\xi \in M_{R((x,y))}$, if ξ' denotes the restriction of ξ to $R((x))(y)$, then the image of the set $\lambda^{-1}(\xi)$ under restriction is precisely the set $\lambda^{-1}(\xi')$.

We know that $\text{Sper}\, R((x))(y) = \text{Sper}\, R_1(y) \,\dot{\cup}\, \text{Sper}\, R_2(y)$. It follows that any $\mathbb{R}$-place of $R((x))(y)$ is the restriction of some $\mathbb{R}$-place of the field $R_k(y)$, for $k \in \{1,2\}$. We use the notation and results of [14] to describe the relationship between orderings and $\mathbb{R}$-places of the field $R((x))(y)$.

The field $F := R((x))$ has exactly two orderings. Fix one of them, and let $\overline{F}$ be the real closure of F at this ordering, so $\overline{F} = R_k$, $k \in \{1,2\}$, and let V and κ be the associated value group and residue field of F. Note that $V = \mathbb{Z} \times V_0$ (lexicographic product) where V_0 is the value group of R, and $\kappa =$ the residue field of R. The value group and residue field of $\overline{F}$ are $\overline{V} = \mathbb{Q} \times V_0$ and $\overline{\kappa} = \kappa$. Let P be a fixed ordering of $\overline{F}(y)$, let $F' := F(y) = R((x))(y)$, and let V' and κ' the associated value group and residue field of F'. Let ξ be the $\mathbb{R}$-place determined by P. By the Baer-Krull Theorem, there are exactly $[V' : 2V']$ orderings on F' having $\mathbb{R}$-place equal to ξ.

Fix a proper truncation closed embedding $p_0 : R \hookrightarrow \kappa((V_0))$. Such an embedding always exists [14] [18]. Consider the embedding $p_k : \overline{F} \hookrightarrow \kappa((\overline{V}))$, defined by $\sum_i a_i x^i \mapsto \sum_{i,j} a_{ij} x^{(i,j)}$ if $k=1$, $\sum_i a_i(-x)^i \mapsto \sum_{i,j} a_{ij} x^{(i,j)}$ if $k=2$, where the a_{ij} are defined by $p_0(a_i) = \sum_j a_{ij} x^j$. This is proper truncation closed and satisfies $p_k(F) \subseteq \kappa((V))$. According to [14, Theorem 1.1], P determines a canonical element $\phi \in \overline{\kappa'}((V'))$, and an extension of p_k to an order preserving embedding $p : \overline{F}(y) \hookrightarrow \overline{\kappa'}((\overline{V'}))$ given by $y \mapsto \phi$. The group V' is generated by V and the support of ϕ. The field κ' is the subfield of $\mathbb{R}$ generated by κ and the coefficients of ϕ.

For any character χ of $V'/2V'$, the map $\sum a_\delta x^\delta \mapsto \sum a_\delta (-1)^{\chi(\delta+2V')} x^\delta$ defines an automorphism t_χ of the field $\overline{\kappa'}((V'))$. The composite embedding $t_\chi \circ p : F(y) \to \overline{\kappa'}((\overline{V'}))$ induces an ordering on $F(y)$. The canonical element of $\overline{\kappa'}((V'))$ determined by this ordering is $t_\chi(\phi)$. The restriction of $t_\chi \circ p$ to F is either p_1 or p_2. (It is p_k iff $\chi((1,0)+2V') = 0$.) The orderings on $F(y)$ defined by the composite embeddings

$t_\chi \circ p$, $\chi \in \chi(V'/2V')$, are distinct and have the same $\mathbb{R}$-place as P. All orderings on $F(y)$ having the same $\mathbb{R}$-place as P are obtained in this way, as χ runs through the character group $\chi(V'/2V')$.

As explained in [14, Theorem 1.1] there are three cases to consider:
(1) immediate transcendental case;
(2) residue transcendental case;
(3) value transcendental case.
We apply [14, Theorem 5.1], bearing in mind that $V = \mathbb{Z} \times V_0$ where V_0 is divisible, and κ is real closed. In case (1) V'/V is countable (but note that V'/V can be finite only in the case when R is non-archimedean) and $\kappa' = \kappa$. In case (2) V'/V is finite and κ' is purely transcendental over κ of transcendence degree 1. Case (2) cannot occur if $\mathbb{R} \subseteq R$. In case (3) $V' = W \oplus \mathbb{Z}\delta$ where $\mathbb{Z}\delta$ is infinite cyclic, $W \supseteq V$, W/V finite, and $\kappa' = \kappa$.

Theorem 6.4. *The index $[V' : 2V']$ is either 1, 2 or 4. In case (1) $[V' : 2V'] = 1$ or 2 depending on whether or not V' is 2-divisible. In case (2) $V' = \frac{1}{d}\mathbb{Z} \times V_0$, $d \geq 1$ and $[V' : 2V'] = 2$. In case (3) $W = \frac{1}{d}\mathbb{Z} \times V_0$, $d \geq 1$ and $[V' : 2V'] = 4$.*

In the terminology of [14, Theorem 1.1], ϕ is distinguished. It has the form w, $w + ax^\gamma$, or $w \pm x^\gamma$, depending on which of the three cases one is considering. Here $w = \sum w_\delta x^\delta$, an element of $\kappa((\overline{V}))$. In case (1), $\phi = w$, $w \notin p(\overline{F})$ but every proper truncation of w is in $p(\overline{F})$. In case (2), $\phi = w + ax^\gamma$, $\gamma \in \overline{V}$, $a \in \mathbb{R}\backslash\kappa$, $w \in p(\overline{F})$ and $w_\delta = 0$ for all $\delta \geq \gamma$. In case (3), $\phi = w \pm x^\gamma$, $\gamma \notin \overline{V}$, $w \in p(\overline{F})$ and $w_\delta = 0$ for all $\delta > \gamma$. It is a straightforward matter to write down formulas for the characters of the group $V'/2V'$ in each of the three cases, and also to write down formulas for each of the power series $t_\chi(\phi)$, $\chi \in \chi(V'/2V')$. In this way, everything we have done here can be made very explicit.

There is an obvious sufficient condition, expressible in terms of the underlying cyclic 2-structure (S, Φ), for two orderings P and Q to have the same associated $\mathbb{R}$-place. In our next theorem we prove that, in the archimedean case, this sufficient condition is also necessary. This is a nice result, but the proof is rather involved, as there are many cases and subcases to consider.

Theorem 6.5. *Let P and Q be two distinct orderings of $R((x))(y)$ or of $R((x,y))$.*

(1) A sufficient condition for P and Q to have the same associated $\mathbb{R}$-place is that for each pair of intervals (r_1, s_1) and (r_2, s_2) of the cyclically ordered set S with $r_1 < P < s_1$ and $r_2 < Q < s_2$, there exists $r \in S$ such that $r_1 < r < s_1$ and $r_2 < r' < s_2$.

(2) If the real closed field R is archimedean then the sufficient condition described in (1) is also necessary.

Proof. It suffices to give the proof for the field $R((x))(y)$.

(1) This is more or less clear. Suppose $\lambda(P) \neq \lambda(Q)$. Using the continuity of λ plus the fact that the space of $\mathbb{R}$-places is hausdorff, there exist open sets U_1 and U_2 in $\operatorname{Sper} R((x))(y)$ with $P \in U_1$, $Q \in U_2$ and $\lambda(U_1) \cap \lambda(U_2) = \emptyset$. Replacing U_1 and U_2 by smaller open sets if necessary, we may assume U_i is defined by some interval (r_i, s_i) in S, for $i = 1, 2$. For any $r \in S$, the principal cuts r_-, r_+, r'_-, r'_+ have the same $\mathbb{R}$-place so we must have $\{r_-, r_+, r'_-, r'_+\} \cap U_i = \emptyset$, for $i = 1$ or 2. It follows that there does not exist $r \in S$ such that $r_1 < r < s_1$ and $r_2 < r' < s_2$.

(2) Suppose now that R is archimedean. Thus $\kappa = R$, $V_0 = \{0\}$, $V = \mathbb{Z}$ and $\overline{V} = \mathbb{Q}$. Suppose $\lambda(P) = \lambda(Q)$ and r_i, s_i are given, $i = 1, 2$, such that $r_1 < P < s_1$ and $r_2 < Q < s_2$.

Immediate transcendental case. Suppose the embedding corresponding to P is given by $x \mapsto x$, $y \mapsto w$, $w = \sum w_\delta x^\delta \in R((\mathbb{Q}))$. The other case, where the embedding corresponding to P is given by $-x \mapsto x$, $y \mapsto w$ is similar and will be omitted. Since Q has the same $\mathbb{R}$-place as P and $Q \neq P$, $(V' : 2V') = 2$. We know that V' is generated over $\mathbb{Z}$ by the exponents of the x^δ appearing in w, so there is some highest 2-power, say it is 2^ℓ, dividing the denominators of the exponents of the x^δ appearing in w. Thus w has the form $w = \sum w_{a/b} x^{a/2^\ell b}$, with $a, b \in \mathbb{Z}$, some a odd, all b odd. Computing $t_\chi(w)$ for the non-trivial character χ of $V'/2V'$, we see that $t_\chi(w) = \sum w_{a/b}(-1)^a x^{a/2^\ell b}$. The embedding corresponding to Q is given by $(-1)^{2^\ell} x \mapsto x$, $y \mapsto \sum w_{a/b}(-1)^a x^{a/2^\ell b}$. There are two cases depending on whether 2^ℓ is even (i.e., $\ell \geq 1$) or 2^ℓ is odd (i.e., $\ell = 0$). In either case any sufficiently fine proper truncation r of w satisfies $r_1 < r < s_1$ and $r_2 < r' < s_2$. (We remark that any proper truncation of w has just finitely many terms.)

Residue transcendental case. Suppose the embedding corresponding to P is given by $x \mapsto x$, $y \mapsto w + ax^\gamma$. The other case, where the embedding corresponding to P is given by $-x \mapsto x$, $y \mapsto w + ax^\gamma$ is similar and will be omitted. We know that V' is generated over $\mathbb{Z}$ by γ and the exponents appearing in w. (Note that the series w has just finitely many terms.) Q is the ordering determined by the embedding $(-1)^d x \mapsto x$, $y \mapsto t_\chi(w + ax^\gamma)$ where d is defined by $V' = \frac{1}{d}\mathbb{Z}$ and χ is the non-trivial character of $V'/2V'$. There are two cases, depending on whether d is even or odd. Pick r of the form $r = w + a_1 x^\gamma$, $a_1 \in R$. The point is that, in either case, if we choose a_1 sufficiently close to a then $r_1 < r < s_1$ and $r_2 < r' < s_2$.

Value transcendental case. The embedding corresponding to P has the form $\pm x \mapsto x$, $y \mapsto w \pm x^\gamma$, so there are four cases to consider. We consider only the case $x \mapsto x$, $y \mapsto w + x^\gamma$. The other cases are dealt with similarly. $V' = \frac{1}{d}\mathbb{Z} \oplus \mathbb{Z}\gamma$ for some integer $d \geq 1$ and $(V' : 2V') = 4$. d is the least common denominator of the exponents of w, and w is expressible in the form $w = \sum w_i x^{i/d}$. The embedding corresponding to Q is given by $x \mapsto x$, $y \mapsto w - x^\gamma$ or $(-1)^d x \mapsto x$, $y \mapsto \sum w_i(-1)^i x^{i/d} + x^\gamma$ or $(-1)^d x \mapsto x$, $y \mapsto \sum w_i(-1)^i x^{i/d} - x^\gamma$.

Fix an integer ℓ and take r of the form $r = w + x^{\alpha/2^\ell \beta}$ where α, β are odd integers ≥ 1. We claim that for an appropriate choice of ℓ and for $\alpha/2^\ell\beta$ is sufficiently close to γ, $r \in (r_1, s_1)$ and $r' \in (r_2, s_2)$. The choice of ℓ depends on which case we are considering. Let 2^m be the highest power of 2 dividing d. If Q is given by $x \mapsto x$, $y \mapsto w - x^\gamma$ choose $\ell = m + 1$, so $r' = w - x^{\alpha/2^\ell\beta}$. If Q is given by $(-1)^d x \mapsto x$ and $y \mapsto \sum(-1)^i w_i x^{i/d} + x^\gamma$, take $\ell = m - 1$, so $r' = \sum(-1)^i w_i x^{i/d} + x^{\alpha/2^\ell\beta}$ if d is even, resp., $r' = \sum(-1)^i w_i(-x)^{i/d} + (-x)^{\alpha/2^\ell\beta}$ if d is odd. If Q is given by $(-1)^d x \mapsto x$ and $y \mapsto \sum(-1)^i w_i x^{i/d} - x^\gamma$ take $\ell = m$, so $r' = \sum(-1)^i w_i x^{i/d} - x^{\alpha/2^\ell\beta}$ if d is even, resp., $r' = \sum(-1)^i w_i(-x)^{i/d} - (-x)^{\alpha/2^\ell\beta}$ if d is odd. □

If the real closed field R is not archimedean then the sufficient condition given in part (1) of Theorem 6.5 is not necessary.

Example 6.6. We know $V = \mathbb{Z} \times V_0$ ordered lexicographically. If R is not archimedean then $V_0 \neq \{0\}$. Fix a proper cut (A, B) of V_0 and take $\gamma = (1, \gamma_0)$ where $A < \gamma_0 < B$. Consider the orderings P and Q of $R((x, y))$ corresponding to the embeddings $x \mapsto x$, $y \mapsto x^{1/2} + x^{\gamma}$ and $x \mapsto x$, $y \mapsto x^{1/2} - x^{\gamma}$ respectively. Clearly $\lambda(P) = \lambda(Q)$. Any $r \in R_1$ close to P has the form $r = x^{1/2} + ax + \cdots$ for some $a \in R$, $a > 0$. Then r' has the form $r' = x^{1/2}+ax+\cdots$ or $r' = -x^{1/2}+ax+\cdots$. In either case, r' is not close to Q.

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