POSITIVITY, SUMS OF SQUARES AND THE MULTI-DIMENSIONAL MOMENT PROBLEM

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Contents

Let K be the basic closed semi-algebraic set in $\mathbb{R}^n$ defined by some finite set of polynomial inequalities $g_1 \geq 0, \ldots, g_s \geq 0$ and let T be the preordering in the polynomial ring $\mathbb{R}[X] := \mathbb{R}[X_1, \ldots, X_n]$ generated by $g_1, \ldots, g_s$. For K compact, Schmüdgen proves in [10] that:

$(*)$ The K-Moment Problem has a positive solution.

$(\dagger)$ $\forall\ f \in \mathbb{R}[X],\ f \geq 0$ on $K \Rightarrow \forall$ real $\epsilon > 0,\ f + \epsilon \in T$.

In the present paper, we consider the status of $(*)$ and $(\dagger)$ when K is not compact. At the same time, we consider a third property:

$(\ddagger)$ $\forall\ f \in \mathbb{R}[X],\ f \geq 0$ on $K \Rightarrow \exists\ q \in T$ such that $\forall$ real $\epsilon > 0,\ f + \epsilon q \in T$

which we prove is strictly weaker than $(\dagger)$ and, at the same time, which implies $(*)$. Many (non-compact) examples are given where $(\ddagger)$ holds. Many examples are given where $(*)$ fails. The question of whether or not $(\ddagger)$ and $(*)$ are equivalent is an open problem.

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In Section 1 we introduce the problem and the notation. In Section 2 we settle the case $n = 1$. In Section 3 we study property $(*)$ using the method developed by Berg, Christensen and Jensen in [1]. We combine this with a result of Scheiderer in [9] to prove that $(*)$ fails whenever K contains a cone of dimension 2. In Section 4 we mention applications of results of the second Author in [7]. In particular, we construct a large number of non-compact examples where $(\ddagger)$ holds. In Section 5 we prove $(\ddagger)$ holds for cylinders with compact cross-section. In Section 6 we provide a list of open problems.

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1. Introduction

Fix an integer $n \geq 1$ and denote the polynomial ring $\mathbb{R}[X_1, \ldots, X_n]$ by $\mathbb{R}[X]$ for short. $\sum \mathbb{R}[X]^2$ denotes the set of finite sums $\sum f_i^2$, $f_i \in \mathbb{R}[X]$. For $S = \{g_1, \ldots, g_s\}$, a finite subset of $\mathbb{R}[X]$, let K_S denote the basic closed semi-algebraic set in $\mathbb{R}^n$ determined by S; i.e.,

$$K_S = \{x \in \mathbb{R}^n \mid g_i(x) \geq 0, i = 1, \ldots, s\}.$$

Let T_S denote the preordering in $\mathbb{R}[X]$ generated by S, i.e., the set of all sums $\sum \sigma_e g^e$, $e = (e_1, \ldots, e_s)$ running through the finite set $\{0,1\}^s$, $\sigma_e \in \sum \mathbb{R}[X]^2$, where $g^e := g_1^{e_1} \ldots g_s^{e_s}$.

Denote by T_S^{alg} the set $\{f \in \mathbb{R}[X] \mid f \geq 0 \text{ on } K_S\}$. Points x in $\mathbb{R}^n$ are in one-to-one correspondence with algebra homomorphisms $L : \mathbb{R}[X] \to \mathbb{R}$ via

$$x = (L(X_1), \ldots, L(X_n)),\ L(f) = f(x).$$

Under this correspondence, points x in K_S correspond to algebra homomorphisms L satisfying $L(g_i) \geq 0$, $i = 1, \ldots, s$ or, equivalently, $L(T_S) \geq 0$.

One can also consider the dual cone

$$K_S^{\text{lin}} = \{L : \mathbb{R}[X] \to \mathbb{R} \mid L \text{ is linear } (\neq 0) \text{ and } L(T_S) \geq 0\},$$

and the double dual cone $T_S^{\text{lin}} = \{f \in \mathbb{R}[X] \mid L(f) \geq 0 \text{ for all } L \in K_S^{\text{lin}}\}$; see Section 3 for a more thorough discussion of the double dual cone. Since every algebra homomorphism is, in particular, a linear map, we have that

$$K_S \hookrightarrow K_S^{\text{lin}} \text{ and } T_S^{\text{alg}} \supseteq T_S^{\text{lin}} \supseteq T_S.$$

Note: T_S, T_S^{lin} generally depend on S. T_S^{alg} depends only on the basic closed set K_S.

Note: $L \in K_S^{\text{lin}} \Rightarrow L(1) > 0$. For $1 \in T_S$ implies $L(1) \geq 0$. If $L(1) = 0$ then, for any $f \in \mathbb{R}[X]$, the identity $(k \pm f)^2 = k^2 \pm 2kf + f^2$ implies $\pm 2kL(f) + L(f^2) \geq 0$ for all real k, so $L(f) = 0$.

We are interested here in the relationship between T_S^{alg}, T_S^{lin} and T_S for all the various choices of (finite) S in $\mathbb{R}[X]$, $n \geq 1$. In particular, we are interested in the relationship between the condition:

$$(*) \qquad T_S^{\text{alg}} = T_S^{\text{lin}}$$

and the condition:

$$(\dagger) \qquad \forall f \in \mathbb{R}[X], f \in T_S^{\text{alg}} \Rightarrow \forall \text{ real } \epsilon > 0, f + \epsilon \in T_S.$$

We are also interested in weak version of ($\dagger$) referred to as ($\ddagger$) (see below) and a strong version of ($\dagger$); namely $T_S^{\text{alg}} = T_S$. The condition $T_S^{\text{alg}} = T_S$ is examined in the recent paper of Scheiderer [9].

The study of the relationship between T_S^{alg} and T_S goes back at least to Hilbert and is a cornerstone of modern semi-algebraic geometry. The Positivstellensatz proved by Stengle in 1974 implies, in particular, that

$$\forall f \in \mathbb{R}[X], f \in T_S^{\text{alg}} \Leftrightarrow \exists p, q \in T_S \text{ and } m \geq 0 \text{ such that } pf = f^{2m} + q.$$

The interest in T_S^{lin} comes from functional analysis. For a linear functional L on $\mathbb{R}[X]$, one is interested in when there exists a positive Borel measure μ on $\mathbb{R}^n$ supported by some given closed set K in $\mathbb{R}^n$ such that

$$\forall f \in \mathbb{R}[X], L(f) = \int_{\mathbb{R}^n} f d\mu.$$

According to a result of Haviland [4] [5], this will be the case iff $L(f) \geq 0$ holds for all $f \in T_S^{\text{alg}}$. The Moment Problem (or at least a version of it) is the following: When is it true that every $L \in K_S^{\text{lin}}$ comes from a positive Borel measure μ on $\mathbb{R}^n$ supported by K_S? In view of the result of Haviland, this is equivalent to asking: When does ($*$) hold?

1.1 Remark. For each n-tuple of non-negative integers $k = (k_1, \dots, k_n)$, denote the monomial $X_1^{k_1} \dots X_n^{k_n}$ by X^k for short. The monomials form a basis for $\mathbb{R}[X]$ so each linear functional L on $\mathbb{R}[X]$ corresponds to a function $p : (\mathbb{Z}^+)^n \to \mathbb{R}$ via $p(k) = L(X^k)$. We say p is positive definite if

$$\sum_{i,j=1}^{m} p(k_i + k_j) c_i c_j \geq 0$$

for arbitrary (distinct) $k_1, \dots, k_m \in (\mathbb{Z}^+)^n$, $c_1, \dots, c_m \in \mathbb{R}$, and $m \geq 1$. For $g \in \mathbb{R}[X]$, $g = \sum a_k X^k$, $g(E)p : (\mathbb{Z}^+)^n \to \mathbb{R}$ is defined by

$$g(E)p(\ell) = \sum_k a_k p(k + \ell).$$

The condition that $L \in K_S^{\mathrm{lin}}$ corresponds exactly to the condition that functions $g^e(E)p$, $e \in \{0,1\}^s$ are positive definite, see [8] or [10]. Consequently, $(*)$ is equivalent to the assertion that every non-zero function $p : (\mathbb{Z}^+)^n \to \mathbb{R}$ with $g^e(E)p$ positive definite for all $e \in \{0,1\}^s$ comes from a positive Borel measure on $\mathbb{R}^n$ supported by K_S.

Schmüdgen's 1991 paper [10] settles the Moment Problem in the compact case. In [10], Schmüdgen proves if K_S is compact then $(*)$ and $(\dagger)$ both hold. In [11], Wörmann proves K_S is compact iff T_S is archimedean and uses this to obtain Schmüdgen's result as a corollary of the Kadison-Dubois Theorem.

Denote by P_d the (finite dimensional) vector space consisting of all polynomials in $\mathbb{R}[X]$ of degree $\leq 2d$, and let $T_d = T_S \cap P_d$. T_d is obviously a cone in P_d, i.e., $T_d + T_d \subseteq T_d$ and $\mathbb{R}^+ T_d \subseteq T_d$. Denote by $\overline{T}_d$ (resp., $\mathrm{int}(T_d)$) the closure (resp., interior) of T_d in P_d. The identity $pq = \frac{1}{2}((p+q)^2 - p^2 - q^2)$ applied to monomials p, q of degree $\leq d$ implies $P_d = T_d - T_d$. This means that T_d contains a basis of P_d, so $\mathrm{int}(T_d) \neq \emptyset$.

1.2 Remark. T_S is archimedean iff $1 \in \mathrm{int}(T_d)$ for all $d \geq 0$. This is just a matter of sorting out the definitions: If $k \pm f \in T_S$ for sufficiently large $k \in \mathbb{Z}$, then $1 \pm \epsilon f \in T_S$ for sufficiently small $\epsilon > 0$. In fact, one even knows that T_S is archimedean iff $k - \sum_{i=1}^n X_i^2 \in T_S$ for k sufficiently large [11], so T_S is archimedean iff $1 \in \mathrm{int}(T_1)$.

We are also interested in the condition:

$(\ddagger)$ $\quad \forall f \in \mathbb{R}[X], f \in T_S^{\mathrm{alg}} \Rightarrow \exists q \in T_S$ such that $\forall$ real $\epsilon > 0, f + \epsilon q \in T_S$.

Obviously, $(\dagger) \Rightarrow (\ddagger)$ (taking $q = 1$). The significance of $(\ddagger)$ is clear from the following:

1.3 Proposition. *The following are equivalent:*

(1) *(‡) holds.*
(2) $T_S^{\mathrm{alg}} = T_S^{\mathrm{lin}} = \cup_{d\geq 0}\overline{T}_d$.

Proof. Clearly $\overline{T}_d \subseteq T_S^{\mathrm{lin}}$. For $f \in P_d$, $q \in T_d$, the condition that $f + \epsilon q \in T_S$ for all real $\epsilon > 0$ is equivalent to:

$$\lambda f + (1-\lambda)q \in T_d \text{ for every } 1 < \lambda < 1,$$

i.e., that every point on the open line segment joining q and f belongs to T_d. In particular, it implies that $f \in \overline{T}_d$. Thus we see that (1) $\Rightarrow$ (2). Now assume (2), and suppose $f \in T_S^{\mathrm{alg}}$. Thus, for d sufficiently large, $f \in \overline{T}_d$. Pick q any point in the interior of T_d. (Recall: $\mathrm{int}(T_d) \neq \emptyset$.) Then obviously the open line segment joining f with q belongs to T_d, so (1) holds. $\square$

Thus, condition (‡) implies not only that (∗) holds, but also that $T_S^{\mathrm{lin}} = \cup_{d\geq 0}\overline{T}_d$.

Note: The requirement in (‡) that q belongs to T_S is unnecessary: Using

$$q = (\frac{q+1}{2})^2 - (\frac{q-1}{2})^2,$$

we see that if $f + \epsilon q \in T_S$, $q \in \mathbb{R}[X]$, then $f + \epsilon(\frac{q+1}{2})^2 \in T_S$.

See Section 4 for additional discussion of (‡) and for examples where (‡) holds. See Section 5 for examples where (‡) holds but (†) does not hold. No examples are known where (∗) holds but (‡) does not hold. No examples are known where $T_S^{\mathrm{lin}} \neq \cup_{d\geq 0}\overline{T}_d$. In fact, there is a general shortage of examples.

Instead of working with T_S, one can work with the $\sum \mathbb{R}[X]^2$-module M_S generated by S, i.e., the set of all sums $\sigma_0 + \sum_{i=1}^s \sigma_i g_i$, $\sigma_i \in \sum \mathbb{R}[X]^2$, $i = 0, \dots, s$, and M_S^{lin}, the double dual cone of M_S. One is interested in the analog of (∗):

$$(*_M) \qquad T_S^{\mathrm{alg}} = M_S^{\mathrm{lin}},$$

the analog to (†):

$$(\dagger_M) \qquad \forall f \in \mathbb{R}[X], f \geq 0 \text{ on } K_S \Rightarrow \forall \text{ real } \epsilon > 0, f + \epsilon \in M_S,$$

and the analog to (‡):

$$(\ddagger_M) \qquad \forall f \in \mathbb{R}[X], f \in T_S^{\mathrm{alg}} \Rightarrow \exists q \in M_S \text{ such that } \forall \text{ real } \epsilon > 0, f + \epsilon q \in M_S.$$

Again, $T_S^{\text{alg}} = M_S$ implies $(\dagger_M)$ which, in turn, implies $(\ddagger_M)$ and $(\ddagger_M)$ is equivalent to $T_S^{\text{alg}} = M_S^{\text{lin}} = \cup_{d\geq 0}\overline{M}_d$ where $\overline{M}_d$ denotes the closure in P_d of $M_d := M_S \cap P_d$. $(\dagger_M)$ is known to hold in a few cases, for example, if K_S is compact and either $n = 1$ [6] or the polynomials $g_1, \dots, g_s$ are linear [6] [8] or $s \leq 2$ [6]. In [6], Jacobi and Prestel develop a powerful valuation-theoretic method for testing the validity of $(\dagger_M)$ when K_S is compact.

2. The case $n = 1$

In this section we consider the case $n = 1$, extending the work of Berg and Maserick in [3].

2.1 Theorem. *Suppose $n = 1$. If K_S is compact, then $(\dagger)$ and $(\dagger_M)$ hold. If K_S is not compact, then the conditions $(\dagger)$, $(\ddagger)$, $(*)$ and $T_S^{\text{alg}} = T_S$ are equivalent and the conditions $(\dagger_M)$, $(\ddagger_M)$, $(*_M)$ and $T_S^{\text{alg}} = M_S$ are equivalent.*

Proof. If K_S is compact then $(\dagger)$ holds by Schmüdgen's result and $(\dagger_M)$ holds by [6, Remark 4.7]. We defer the proof of the rest of Theorem 2.1 to Section 3 where it is an immediate consequence of Theorem 3.5. □

Note: Already in the case $n = 1$, we see the dichotomy between the compact case and the non-compact case. If K_S is compact then $(\dagger)$ (resp., $(\dagger_M)$) holds but, unlike what happens in the non-compact case, this does not imply that $T_S^{\text{alg}} = T_S$ (resp., $T_S^{\text{alg}} = M_S$) holds. For example, if $S = \{(1 - X^2)^3\}$ then $T_S^{\text{alg}} \neq T_S$; see [8, Example 4.3.3].

The second part of Theorem 2.1 is only useful if we know when $T_S^{\text{alg}} = T_S$ (resp., $T_S^{\text{alg}} = M_S$) holds. For the rest of the section we concentrate our attention on answering this question.

2.2 Theorem. *Suppose $n = 1$. If K_S is not compact and $T_S^{\text{alg}} = T_S$, then:*

- (i) *If K_S has a smallest element, call it a, then $r(X - a) \in S$ for some real $r > 0$.*
- (ii) *If K_S has a largest element, call it a, then $r(a - X) \in S$ for some real $r > 0$.*
- (iii) *For every $a, b \in K_S$ with $a < b$ and $(a, b) \cap K_S = \emptyset$, $r(X - a)(X - b) \in S$ for some real $r > 0$.*

Conversely, for any K_S, if conditions (i), (ii) and (iii) hold, then $T_S^{\text{alg}} = T_S$.

2.3 Notes.

(1) Since K_S is a closed semi-algebraic set in $\mathbb{R}$, it is the union of finitely many closed intervals and points (including possibly closed intervals of the form $(-\infty, a]$ or $[a, \infty)$).

(2) If $K \subseteq \mathbb{R}$ is any closed semi-algebraic set then one checks easily that $K = K_S$ for S the set of polynomials defined as follows:

— If $a \in K$ and $(-\infty, a) \cap K = \emptyset$, then $X - a \in S$.

— If $a \in K$ and $(a, \infty) \cap K = \emptyset$, then $a - X \in S$.

— If $a, b \in K$, $a < b$, $(a, b) \cap K = \emptyset$, then $(X - a)(X - b) \in S$.

— S has no other elements except these.

We call S the natural choice of generators for K.

(3) Theorem 2.2 proves that, for $K = K_S$ not compact, $T_S^{\text{alg}} = T_S$ holds iff S contains the natural choice of generators (up to scalings by positive reals).

(4) If K_S is compact then $T_S^{\text{alg}} = T_S$ can hold without conditions (i), (ii) and (iii) holding. For example, if $S = \{1 - X^2\}$, then $1 \pm X = \frac{1}{2}(1 \pm X)^2 + \frac{1}{2}(1 - X^2) \in T_S$ so $T_S^{\text{alg}} = T_S$.

Proof. Suppose $T_S^{\text{alg}} = T_S$, K_S not compact. We can assume that the elements of S have degree ≥ 1. Since K_S is not compact, it either contains an interval of the form $[a, \infty)$ or it contains an interval of the form $(-\infty, a]$. Replacing X by $-X$ if necessary, we can assume we are in the first case. Thus each $g_i \in S$ is non-negative on some interval $[a, \infty)$, so has positive leading coefficient. Thus, for any element $p = \sum \sigma_e g_1^{e_1} \ldots g_s^{e_s}$ of T_S, the degree of p is equal to the maximum of the degrees of the terms $\sigma_e g_1^{e_1} \ldots g_s^{e_s}$. Also, since each σ_e is a sum of squares, it has even degree.

Suppose that K_S has a smallest element, a say. Let $p = X - a$. Then $p \geq 0$ on K_S so $p \in T_S$. Since p has degree 1, p is a sum of terms of the form σ, $\sigma \in \mathbb{R}^+$, and σg_i with $\sigma \in \mathbb{R}^+$ and $g_i \in S$ linear. Each such g_i is ≥ 0 at a (since $a \in K_S$). Consequently, at least one linear g_i in S is equal to zero at a, so $g_i = r(X - a)$ as required.

Suppose now that $a, b \in K_S$ are such that $a < b$ and $(a, b) \cap K_S = \emptyset$. Let $p = (X - a)(X - b)$. Then $p \geq 0$ on K_S so $p \in T_S$. Since p has degree 2, it is a sum of terms of the form $\sigma \in \sum \mathbb{R}[X]^2$ of degree 0 or 2, σg_i with $\sigma \in \mathbb{R}^+$ and $g_i \in S$ linear or quadratic, and $\sigma g_i g_j$ with $\sigma \in \mathbb{R}^+$ and $g_i, g_j \in S$ linear. Since any linear $g_i \in S$ is increasing and $g_i(a) \geq 0$, g_i is positive on the interval (a, b). Thus $p \geq \sigma_1 g_1 + \cdots + \sigma_t g_t$ on (a, b) where $g_1, \ldots, g_t$ are quadratics in S which assume at least one negative value on (a, b), and $\sigma_1, \ldots, \sigma_t$ are positive reals. Define the width of a quadratic g to be $r_2 - r_1$ where $r_1 \leq r_2$ are the roots of g (or 0 if g has no roots). Each g_i opens upward and is non-negative at a, b (since $a, b \in K_S$), has its roots between a and b and consequently has width at most $b - a$, $i = 1, \dots, t$. It suffices to show that g_i has width exactly $b - a$ for some $i \in \{1, \dots, t\}$, for then g_i necessarily has the form $r(X - a)(X - b)$ for some real $r > 0$. Since the width of p is equal to $b - a$ and $\sigma_i g_i$ has the same width as g_i, this is a consequence of the following elementary result:

2.4 Lemma. *If f_1, f_2 are quadratics with positive leading coefficients, then*

$$\mathit{width}(f_1 + f_2) \leq \max\{\mathit{width}(f_1), \mathit{width}(f_2)\}.$$

Proof. Without loss of generality we can assume $\mathrm{width}(f_1) \geq \mathrm{width}(f_2)$ and that f_1 has positive width. Shifting and scaling, we can assume $f_1 = X^2 - X$, $f_2 = c(X - a)(X - (a+b))$, $c > 0$, $0 \leq b \leq 1$. Thus $f_1 + f_2 = (c+1)X^2 - (2ac + bc + 1)X + ca(a+b)$ and

$$\mathrm{width}(f_1 + f_2) = \frac{\sqrt{(2ac + bc + 1)^2 - 4(c+1)ca(a+b)}}{c+1}.$$

Thus we are reduced to showing that

$$\frac{(2ac + bc + 1)^2 - 4(c+1)ca(a+b)}{(c+1)^2} \leq 1, \tag{1}$$

i.e., that

$$(2ac + bc + 1)^2 \leq (c+1)^2 + 4(c+1)ca(a+b). \tag{2}$$

Expanding and canceling, this reduces to showing that

$$b^2c + 4a + 2b \leq c + 2 + 4a^2 + 4ab \tag{3}$$

or, equivalently, that

$$(1 - b^2)(c+1) + (2a + b - 1)^2 \geq 0. \tag{4}$$

Since $0 \leq b \leq 1$ and $c > 0$, this is clear. $\square$

Note: Equality holds iff $b = 1$ and $a = 0$, i.e., iff $f_2 = cf_1$.

Suppose now that (i) (ii) (iii) hold. If K_S has a smallest element a then $X - a \in T_S$ so $X - d = (X - a) + (a - d) \in T_S$ for any $d \leq a$. Similarly, if K_S has a largest element a then $a - X \in T_S$, so $d - X = (a - X) + (d - a) \in T_S$ for any $d \geq a$. Also, if $a, b \in K_S$ are such that $a < b$ and $(a, b) \cap K_S = \emptyset$, then $(X - a)(X - b) \in T_S$. Moreover, by the proof of [3, Lemma 4], if $a \leq c \leq d \leq b$ then $(X - c)(X - d)$ is in the preordering generated by $(X - a)(X - b)$, so $(X - c)(X - d) \in T_S$.

Suppose $f \in \mathbb{R}[X]$, $f \geq 0$ on K_S. We prove $f \in T_S$ by induction on the degree. If f has degree zero it is clear. If $f \geq 0$ on $\mathbb{R}$ then $f \in \sum \mathbb{R}[X]^2$ so, in particular, $f \in T_S$. Thus we can assume that $f(c) < 0$ for some c. There are three possibilities: Either K_S has a least element a and $c < a$ or K_S has a largest element a and $c > a$ or there exist $a, b \in K_S$, $a < b$ with $(a, b) \cap K_S = \emptyset$, and $a < c < b$. In the first case f has a least root d in the interval $(c, a]$, $X - d \in T_S$, $f = (X - d)g$ for some $g \in \mathbb{R}[X]$ and one checks that $g \geq 0$ on K_S. In the second case f has a greatest root d in the interval $[a, c)$, $d - X \in T_S$, $f = (d - X)g$ and again $g \geq 0$ on K_S. Similarly, in the third case, f has greatest root d in the interval $[a, c)$ and a least root e in the interval $(c, b]$, $(X - d)(X - e) \in T_S$, $f = (X - d)(X - e)g$ and $g \geq 0$ on K_S. Thus, in any case, the result follows by induction on the degree. □

The question of when $T_S^{\text{alg}} = M_S$ holds for $K = K_S$ not compact is more complicated. Theorem 2.2 provides an obvious necessary condition: S must contain the natural choice of generators (up to scalings by positive reals). But this necessary condition is sufficient only in very special cases:

2.5 Theorem. *Suppose K is not compact and S is the natural choice of generators for K. Then $T_S^{\text{alg}} = M_S$ holds iff either $|S| \leq 1$ or $|S| = 2$ and K has an isolated point.*

Proof. If $|S| \leq 1$, then $M_S = T_S$ so $T_S^{\text{alg}} = M_S$ by Theorem 2.2; also see [3, Theorem 1]. Suppose $|S| = 2$ and K has an isolated point. If this isolated point is the least element of K, then $S = \{g_1, g_2\}$, $g_1 = X - b$, $g_2 = (X - b)(X - c)$, $b < c$. Then one checks that

$$g_1 g_2 = (X - c)^2 g_1 + (c - b) g_2 \in M_S,$$

so $T_S = M_S$. The case where the isolated point is the greatest element of K is similar. The remaining case is where $S = \{g_1, g_2\}$, $g_1 = (X - a)(X - b)$, $g_2 = (X - b)(X - c)$, $a < b < c$. In this last case one checks that

$$g_1 g_2 = \frac{b - a}{c - a}(X - c)^2 g_1 + \frac{c - b}{c - a}(X - a)^2 g_2 \in M_S.$$

Thus, in all cases, $M_S = T_S$ so $T_S^{\text{alg}} = M_S$ by Theorem 2.2.

Suppose now that either $|S| = 2$ and K has no isolated points, or $|S| \geq 3$. Say $S = \{g_1, \dots, g_s\}$. Replacing X by $-X$ if necessary, we can assume that K contains an interval of the form $[a, \infty)$. Reindexing, we can suppose g_1 is either linear of the form $g_1 = X - b$ (corresponding to the case where K has a least element b) or $g_1 = (X-a)(X-b)$, $a < b$ where (a, b) is the left-most gap of K (corresponding to the case where K has no least element) and that $g_2 = (X-c)(X-d)$ where (c, d) is the right-most gap of K. By our hypothesis, $b < c$. We claim that $g_1 g_2 \notin M_S$. Since $g_1 g_2 \geq 0$ on K, this will complete the proof. Suppose, to the contrary, that

$$g_1 g_2 = \sigma_0 + \sigma_1 g_1 + \dots + \sigma_s g_s, \ \sigma_i \in \sum \mathbb{R}[X]^2.$$

Comparing degrees, we see that σ_i has degree 2 or 0 if $i \geq 1$ and σ_0 has degree 4, 2 or 0. Let $g = g_2 - \sigma_1$. Thus $g_1 g = \sigma_0 + \sigma_2 g_2 + \dots + \sigma_s g_s$. Thus $g_1(c) g(c) \geq 0$, $g_1(d) g(d) \geq 0$, so $g(c) \geq 0$, $g(d) \geq 0$. On, the other hand, from $g = g_2 - \sigma_1$, it follows that $g(c) \leq 0$, $g(d) \leq 0$. Since g has degree ≤ 2, this implies $g = 0$, i.e., $g_2 = \sigma_1$, contradicting the fact that g_2 is strictly negative on (c, d). $\square$

3. Examples where $(*)$ fails

In this section we study condition $(*)$ using an extension of the method introduced by Berg, Christensen and Jensen in [1]. A major tool here is the recent work of Scheiderer in [9].

As we have seen in Section 1:

$$T_S^{\mathrm{alg}} = T_S \Rightarrow (\dagger) \ \Rightarrow \ (\ddagger) \ \Rightarrow \ (*).$$

If $T_S^{\mathrm{lin}} = T_S$ we can obviously say more:

3.1 Proposition. *If $T_S^{\mathrm{lin}} = T_S$ then the conditions $(\dagger)$, $(\ddagger)$, $(*)$ and $T_S^{\mathrm{alg}} = T_S$ are equivalent.*

And similarly, for M_S:

3.2 Proposition. *If $M_S^{\mathrm{lin}} = M_S$ then the conditions $(\dagger_M)$, $(\ddagger_M)$, $(*_M)$ and $T_S^{\mathrm{alg}} = M_S$ are equivalent.*

Proof. This is clear. $\square$

We recall results from the theory of locally convex vector spaces. Any vector space V over $\mathbb{R}$ comes equipped with a unique finest locally convex topology [2, Sect. 1.1.9]. For a cone C in V, C^{lin} denotes the double dual cone of C, i.e., the set of all $x \in V$ such that $L(x) \geq 0$ holds for all linear functionals L on V with $L(C) \geq 0$. We need the following two results:

3.3 Lemma. *If C is a cone in a vector space V over $\mathbb{R}$ then*

(1) C^{lin} *is the closure of* C.
(2) $C^{\text{lin}} = C \Leftrightarrow C$ *is closed.*

Proof. According to the Bipolar Theorem [2, Theorem 1.3.6], C^{lin} is the smallest closed convex set in V containing C. Since the closure of a cone is a cone (so, in particular, is convex), (1) is clear. (2) is immediate from (1). □

3.4 Lemma. *Suppose C is a cone in a vector space V over $\mathbb{R}$ and $C_i = C \cap V_i$ where V_i, $i \geq 0$ are finite dimensional subspaces of V with $V_i \subseteq V_{i+1}$ and $V = \cup_{i \geq 0} V_i$. Then the following are equivalent:*

(1) C *is closed in* V.
(2) C_i *is closed in* V_i *for each* $i \geq 0$.

Proof. See [2, Lemma 6.3.3]. □

Our next result provides examples where T_S and M_S are closed.

3.5 Theorem. *If K_S contains a cone of dimension n then T_S and M_S are closed.*

Note: Theorem 3.5 extends [1, Theorem 3], [2, Lemma 6.3.9] and [3, Lemma 3].

Note: Theorem 3.5 applies, in particular, if $n = 1$ and K_S is not compact. Combining this with Lemma 3.3(2) and Propositions 3.1 and 3.2 completes the proof of Theorem 2.1.

Proof. The first assertion follows from the second, replacing S by the set of all products g^e, $e \in \{0,1\}^s$, $e \neq (0,\dots,0)$. By Lemma 3.4, to prove the second assertion, it suffices to show that M_i is closed in P_i, for each $i \geq 0$. Let $S = \{g_1,\dots,g_s\}$. We can assume each g_i is not zero. By our hypothesis, K_S contains a cone C with non-empty interior. Making a linear change in coordinates if necessary, we can assume the vertex of C is at the origin.

For f any non-zero element of M_S, let $f = f_0 + \cdots + f_d$ be its homogeneous decomposition. Observe that, for any $a \in C$ and any $\lambda > 0$,

$$f(\lambda a) = f_0 + \lambda f_1(a) + \cdots + \lambda^d f_d(a).$$

Since $\lambda a \in C \subseteq K_S$ for all positive λ it follows that, if $f_d(a) \neq 0$, then $f_d(a) > 0$.

Now suppose

$$p = \sigma_0 + \sigma_1 g_1 + \cdots + \sigma_s g_s,$$

$\sigma_i \in \sum \mathbb{R}[X]^2$. Let d denote the maximum of the degrees of $\sigma_0, \sigma_1 g_1, \ldots, \sigma_s g_s$. Using the fact that C has non-empty interior, we can pick a point $a \in C$ such that the homogeneous parts of highest degree of the various polynomials in question do not vanish, so the homogeneous part of p having degree d is positive when evaluated at a (so, in particular, it is not zero). To summarize: if p has degree d, then σ_0 has degree $\le d$ and each σ_i, $i \ge 1$ has degree $\le d - \deg(g_i)$. Thus any $p \in M_m$ is expressible as above with $\sigma_i = \sum f_{ij}^2$, $\deg(f_{0j}) \le m$, $\deg(f_{ij}) \le m - \frac{1}{2}\deg(g_i)$, $i \ge 1$. Also, as in [1], we can assume $\sigma_i = \sum_{j=1}^N f_{ij}^2$, where N is the dimension of P_m.

Making an additional linear change of coordinates, we can also assume that the point $(1, \ldots, 1)$ lies in the interior of C and none of the g_i vanish at this point. Thus, for real distinct $a_1, \ldots, a_{2m+1}$ sufficiently close to 1, the set

$$H = \{(a_{j_1}, \ldots, a_{j_{2m+1}}) \mid 1 \le j_k \le 2m+1, k = 1, \ldots, 2m+1\}$$

is contained in C and $g_i(a) > 0$ for each $a \in H$. As explained in [1], two elements of P_m are equal iff they are equal on H, and the topology on P_m coincides with the topology of pointwise convergence on H. Also, for any $a \in H$, $f_{0j}(a)^2 \le \sigma_0(a) \le p(a)$, and $f_{ij}(a)^2 \le \sigma_i(a) \le p(a)/D_i$ where D_i is the minimum of the $g_i(a)$, $a \in H$, if $i \ge 1$. Thus, if p_ℓ is a sequence in M_m converging to some $p \in P_m$ then, as in [1], there exists a subsequence p_{ℓ_k} with the associated $f_{\ell,i,j}$ converging pointwise on H to some $f_{ij} \in P_m$ of the appropriate degree, so $p \in M_m$. $\square$

Theorem 3.5 does not cover the case where K_S is a curve. At the same time T_S is closed for many curves, at least, for the "right" choice of S. We give two examples to illustrate.

3.6 Example. Consider the curve

$$C = \{(x, y) \in \mathbb{R}^2 \mid y = q(x)\}$$

in $\mathbb{R}^2$, $q \in \mathbb{R}[X]$. Then $C = K_S$ where $S = \{-(Y-q)^2\}$. We claim that (†) holds in this choice of S. Suppose $f \ge 0$ on C. Then $f(X,Y) = f(X, q(X)) + h(X,Y)$ with h vanishing on C. Also $f(X, q(X)) \ge 0$ on $\mathbb{R}$ so $f(X, q(X)) \in \sum \mathbb{R}[X]^2$. Thus we are reduced to showing that $\epsilon + h \in T_S$ for each real $\epsilon > 0$. Since $Y - q$ is irreducible, $h = (Y-q)h_1$ for some $h_1 \in \mathbb{R}[X,Y]$, so

$$\epsilon + h = \epsilon\left(1 + \frac{h_1}{2\epsilon}(Y-q)\right)^2 - \frac{h_1^2}{4\epsilon}(Y-q)^2 \in T_S.$$

Note: $Y-q \notin T_S$. If $Y-q = \sigma_0 - \sigma_1(Y-q)^2$, $\sigma_0, \sigma_1 \in \sum \mathbb{R}[X,Y]^2$, then σ_0 vanishes on C so $(Y-q)^2$ divides σ_0. Dividing through by $Y-q$ this implies $Y-q$ divides 1, a contradiction. Thus $T_S^{\text{alg}} \neq T_S$, so T_S is not closed for this choice of S. On the other hand, the curve C is described more naturally as $C = K_S$ where $S = \{Y-q, -(Y-q)\}$, and, for this choice of S, $T_S^{\text{alg}} = T_S$ (so, in particular, T_S is closed). To prove this, just write $h_1 = r^2 - s^2$, $r = (h_1+1)/2$, $s = (h_1-1)/2$, so

$$f = f(X, g(X)) + r^2(Y-q) - s^2(Y-q) \in T_S.$$

3.7 Example. Consider the curve

$$C = \{(x,y) \in \mathbb{R}^2 \mid y^2 = q(x)\}$$

where $q \in \mathbb{R}[X]$ is not a square. Thus $C = K_S$ where $S = \{Y^2 - q, -(Y^2-q)\}$. Assume that C is not compact. Replacing X by $-X$ if necessary, we can assume that $q(x) > 0$ for x sufficiently large (so the leading coefficient of q is positive). We claim that, under this assumption, T_S is closed. Since $T_S = \sum \mathbb{R}[X,Y]^2 + (Y^2-q)$, where (Y^2-q) denotes the ideal generated by Y^2-q, it suffices to prove that $\sum A^2$ is closed in A, where A denotes the coordinate ring of C, i.e.,

$$A = \frac{\mathbb{R}[X,Y]}{(Y^2-q)}.$$

Every $f \in A$ is expressible uniquely as $f = g + h\sqrt{q}$, $g, h \in \mathbb{R}[X]$. Let Q_d denote the subspace of A consisting of all $f = g + h\sqrt{q} \in A$ with $\deg(g)$, $\deg(h) \leq 2d$. According to Lemma 3.4, it suffices to show that $(\sum A^2) \cap Q_d$ is closed in Q_d for each $d \geq 0$. We need certain degree estimates. If $f = g + h\sqrt{q}$, $g, h \in \mathbb{R}[X]$, then $f^2 = (g^2 + h^2 q) + (2gh)\sqrt{q}$. Thus every element of $\sum A^2$ has the form

$$p = \sum_{i=1}^{k} (g_i + h_i\sqrt{q})^2 = \sum_{i=1}^{k} (g_i^2 + h_i^2 q) + 2\sum_{i=1}^{k} g_i h_i \sqrt{q}.$$

Thus, if $p = p_1 + p_2\sqrt{q}$, $p_1, p_2 \in \mathbb{R}[X]$, then

$$\deg(p_1) = \max\{2\deg(g_i), 2\deg(h_i) + \deg(q) \mid 1 \leq i \leq k\}.$$

In particular, if $p \in Q_d$, then $\deg(g_i) \leq d$ and $\deg(h_i) \leq d - \deg(q)/2$, $i = 1, \ldots, k$. With these estimates in hand, the proof that $(\sum A^2) \cap Q^d$ is closed in Q_d follows along the same lines as in the proof of Theorem 3.5 and can safely be left to the reader.

Since T_S is closed, the various conditions (†), (‡), (∗) and $T_S^{\mathrm{alg}} = T_S$ are equivalent. We claim that these conditions hold if $\deg(q) = 1$ or 2 and that they fail if $\deg(q) \geq 3$. The first assertion is elementary; e.g., see [9, Proposition 2.17]. Suppose now that $\deg(q) \geq 3$. If q is not in $\sum \mathbb{R}[X]^2$ then q takes on negative values so there exists a quadratic $p \in \mathbb{R}[X]$ which takes on negative values with $p \geq 0$ on C. If $p \in T_S$ then $p = \sum_{i=1}^k (g_i^2 + h_i^2 q)$ for some $g_i, h_i \in \mathbb{R}[X]$. Then $\sum_{i=1}^k h_i^2 \neq 0$, so $\deg(p) \geq \deg(q)$, a contradiction. If $q \in \sum \mathbb{R}[X]^2$, then $q = r^2 + s^2$, $r, s \in \mathbb{R}[X]$. Take $p = p_1 + Y$ where $p_1 \in \mathbb{R}[X]$ has degree $\leq \max\{\deg(r), \deg(s)\} + 1$ and is such that $p_1 \geq |r| + |s|$ on $\mathbb{R}$. Then

$$p(x, \pm\sqrt{q(x)}) = p_1(x) \pm \sqrt{q(x)} \geq |r(x)| + |s(x)| \pm \sqrt{r(x)^2 + s(x)^2} \geq 0$$

for each $x \in \mathbb{R}$ so $p \geq 0$ on C. If $p \in T_S$, then $p_1 = \sum_{i=1}^k (g_i^2 + h_i^2 q)$, $1 = 2\sum_{i=1}^k g_i h_i$ for some $g_i, h_i \in \mathbb{R}[X]$. Then $\sum h_i^2 \neq 0$, so $\deg(p_1) \geq \deg(q)$. Since $\deg(q) = \max\{2\deg(r), 2\deg(s)\}$, this contradicts the assumption that $\deg(q) \geq 3$.

Note: According to [9, Theorem 3.4], if C is non-singular and $\deg(q) \geq 3$ then the preordering T_S^{alg} is not even finitely generated.

We turn now to the case where $\dim(K_S) \geq 2$. In [1, Theorem 4], Berg, Christensen and Jensen use their weak version of Theorem 3.5 to show that (∗) fails for $S = \emptyset$ (so $K_S = \mathbb{R}^n$, $T_S = \sum \mathbb{R}[X]^2$) if $n \geq 2$. In [2, Theorem 6.3.9] this same method is used to show that (∗) fails for $S = \{X_1, \dots, X_n\}$ if $n \geq 2$. We proceed to generalize these results. In general, for T_S closed, showing (∗) fails is equivalent to showing $\exists\, p \in T_S^{\mathrm{alg}}$, $p \notin T_S$. Such a polynomial always exists when $\dim(K_S)$ is large enough, e.g., the Motzkin polynomial if $S = \emptyset$, $n \geq 2$. The following general result is proved in [9]:

3.8 Theorem. *If $\dim(K_S) \geq 3$ then there exists a polynomial $p \geq 0$ on $\mathbb{R}^n$ such that $p \notin T_S$.*[1]

Proof. See [9, Prop. 6.1]. □

Theorem 3.8 implies, in particular, that (∗) fails whenever $n \geq 3$ and K_S contains a cone of dimension n. Another result in [9] allows us to greatly improve on this:

[1] Apparently this is not true for $\dim(K_S) = 2$. In the "Added to proof" at the end of [9], Scheiderer reports the discovery of a smooth compact surface K_S with $T_S^{\mathrm{alg}} = T_S$.

3.9 Theorem. *If $n = 2$ and K_S contains a 2-dimensional cone, then there exists a polynomial $p \geq 0$ on $\mathbb{R}^2$ such that $p \notin T_S$.*

Proof. See [9, Remark 6.7]. $\square$

Note: The proof of Theorem 3.9 given in [9] is highly non-trivial. It uses deep results on existence of "enough" psd regular functions on non-compact non-rational curves [9, Section 3] in conjunction with an extension theorem for extending such functions to the ambient space [9, Section 5].

3.10 Corollary. *$(*)$ fails whenever $n \geq 2$ and K_S contains a cone of dimension 2.*

Note: We are not claiming now that T_S is closed.

Proof. Changing coordinates, we may assume the cone is given by

$$X_1 \geq 0,\ X_2 \geq 0,\ X_i = 0,\ i \geq 3.$$

Define $S' \subseteq \mathbb{R}[X_1, X_2]$ by $S' = \{g_1', \dots, g_s'\}$ where $g_i' = g_i(X_1, X_2, 0, \dots, 0)$. Thus $K_{S'}$ contains the cone defined by $X_1 \geq 0$, $X_2 \geq 0$ so, by Theorem 3.5, $T_{S'}$ is closed. Use Theorem 3.9 to pick $p = p(X_1, X_2)$ such that $p \geq 0$ on $\mathbb{R}^2$ and $p \notin T_{S'}$. Then $p \in T_S^{\mathrm{alg}}$. We claim that $p \notin T_S^{\mathrm{lin}}$. For suppose $p \in T_S^{\mathrm{lin}}$. Any linear functional L on $\mathbb{R}[X_1, X_2]$ with $L(T_{S'}) \geq 0$ extends to a linear functional L' on $\mathbb{R}[X]$ with $L'(T_S) \geq 0$ via $L'(f) = L(f(X_1, X_2, 0. \dots, 0))$. This is clear. Thus, for any such L, $L(p) = L'(p) \geq 0$, so $p \in T_{S'}^{\mathrm{lin}}$. Since $T_{S'}^{\mathrm{lin}} = T_{S'}$, this contradicts the choice of p. $\square$

The method of Corollary 3.10 applies in other cases as well:

3.11 Example. Take $n = 2$, $S = \{X, 1 - X, Y^3 - X^2\}$. K_S consists of the points $(x, y) \in \mathbb{R}^2$ on the vertical strip $0 \leq x \leq 1$ with $y \geq x^{2/3}$. We claim that $(*)$ fails for S. Let $S' \subseteq \mathbb{R}[Y]$ be defined by $S' = \{Y^3\}$. Then $K_{S'} = [0, \infty)$ so, by Theorem 3.5, $T_{S'}$ is closed. One checks that $Y \notin T_{S'}$. (Degree considerations show that $Y = \sigma_0 + \sigma_1 Y^3$, $\sigma_0, \sigma_1 \in \sum \mathbb{R}[Y]^2$ is not possible.) Clearly $Y \geq 0$ on K_S. We claim $Y \notin T_S^{\mathrm{lin}}$. We argue as in the proof of Corollary 3.10: Every linear functional L on $\mathbb{R}[Y]$ with $L(T_{S'}) \geq 0$ extends to a linear functional L' on $\mathbb{R}[X, Y]$ with $L'(T_S) \geq 0$ via $L'(f) = L(f(0, Y))$. Thus, if $Y \in T_S^{\mathrm{lin}}$ then $L(Y) = L'(Y) \geq 0$ for every such L, so $Y \in T_{S'}^{\mathrm{lin}} = T_{S'}$.

4. Consequences of the results in [7]

In this section we recall results from [7], and mention applications of these results to the problem at hand. The results in [7] extend Schmüdgen's result in the compact case, replacing the assumption that K_S is compact with the assumption that $p \in 1 + T_S$ is chosen so that the coordinate functions $X_1, \dots, X_n$ are bounded on K_S by kp^ℓ for k, ℓ sufficiently large. The method of proof in [7] is a generalization of Wörmann's method in [11].

Note: Such a polynomial p always exists, e.g., $p = 1 + \sum_{i=1}^n X_i^2$. The 'best' choice of p may depend on K_S. For example:

— If K_S is compact, we can take $p = 1$.

— If $n = 2$ and $S = \{X, 1 - X\}$, we can take $p = 1 + Y^2$.

Fixing such a polynomial p, we have:

4.1 Theorem. *For any $f \in \mathbb{R}[X]$ there exist integers k, ℓ such that $kp^\ell \pm f \in T_S$.*

Proof. See [7, Corollary 1.4]. $\square$

This gives better control over the element q appearing in condition (‡). We can choose q to be a power of p if we want. In more detail, we have:

4.2 Corollary. *The following are equivalent:*

(1) *(‡) holds.*
(2) $\forall f \in \mathbb{R}[X]$, $f \in T_S^{\mathrm{alg}} \Rightarrow \exists \ell \geq 0$ *such that* $\forall$ *real* $\epsilon > 0$, $f + \epsilon p^\ell \in T_S$.

Proof. (2) $\Rightarrow$ (1) is clear. For (1) $\Rightarrow$ (2), use Theorem 4.1 to pick k, ℓ so that $kp^\ell - q \in T_S$ and use the identity $f + \epsilon p^\ell = (f + \frac{\epsilon}{k} q) + \frac{\epsilon}{k}(kp^\ell - q)$. $\square$

The second main result in [7] is:

4.3 Theorem. *For any $f \in \mathbb{R}[X]$, the following are equivalent:*

(1) $f \in T_S^{\mathrm{alg}}$.
(2) *For any sufficiently large integer m and all rational $\epsilon > 0$, there exists an integer $\ell \geq 0$ such that $p^\ell(f + \epsilon p^m) \in T_S$.*

Proof. See [7, Corollary 3.1]. $\square$

Let

$$T'_S = \{f \in \mathbb{R}[X] \mid p^\ell f \in T_S \text{ for some } \ell \geq 0\}.$$

For $d \geq 0$, let $T'_d = T'_S \cap P_d$.

4.4 Corollary. $T_S^{\mathrm{alg}} = (T'_S)^{\mathrm{lin}} = \cup_{d\geq 0}\overline{T'_d}$.

Proof. Suppose $f \in T_S^{\mathrm{alg}}$. By Theorem 4.3 there exists $m \geq 0$ such that, for all $\epsilon > 0$, $f + \epsilon p^m \in T'_S$. This implies $f \in \overline{T'_d}$ for d sufficiently large. $\square$

Note: In view of the result of Haviland in [4] [5], Corollary 4.4 implies, in particular, that a linear functional L on $\mathbb{R}[X]$ comes from a positive Borel measure on $\mathbb{R}^n$ supported by K_S iff $L(T'_S) \geq 0$; also see [8].

Note: The preordering T'_S is not finitely generated in general. This is the case, for example, if K_S contains a cone of dimension 2; see Corollary 3.10.

Theorem 4.3 also yields a large number of non-compact examples where (‡) holds. Denote by $\mathbb{R}[X, Y]$ the polynomial ring in $n+1$ variables $X_1, \dots, X_n, Y$ and consider the finite set $S' = S \cup \{1 - pY, -(1 - pY)\}$ in $\mathbb{R}[X, Y]$. $K_{S'}$ consists of those points on the hypersurface

$$H = \{(x, y) \in \mathbb{R}^{n+1} \mid p(x)y = 1\}$$

in $\mathbb{R}^{n+1}$ which map to K_S under the projection $(x, y) \mapsto x$.

4.5 Corollary. *(‡) holds for S'.*

Proof. Suppose $f \in T_{S'}^{\mathrm{alg}}$. Thus $f(X, \frac{1}{p}) \geq 0$ on K_S. Writing $f(X, \frac{1}{p}) = \frac{g}{p^k}$, $g \in \mathbb{R}[X]$, we see that $g \geq 0$ on K_S so there exists $m \geq 0$ such that for each $\epsilon > 0$, $p^\ell(g + \epsilon p^m) \in T_S$ for some $\ell \geq 0$. Since $p - 1 \in T_S$, there is no harm in assuming $m \geq k$. Thus

$$\frac{g}{p^k} + \epsilon p^{m-k} = \frac{h}{p^{k+\ell}} \tag{1}$$

for some $h \in T_S$. This implies that the polynomial $f + \epsilon p^{m-k} - hp^{k+\ell}Y^{2(k+\ell)}$ vanishes on H so

$$f + \epsilon p^{m-k} = hp^{k+\ell}Y^{2(k+\ell)} + j(1 - pY) \tag{2}$$

for some $j \in \mathbb{R}[X, Y]$. Using the identity $j = (\frac{j+1}{2})^2 - (\frac{j-1}{2})^2$ this implies $f + \epsilon p^{m-k} \in T_{S'}$ for all real $\epsilon > 0$. $\square$

Thus, for example, although $(*)$ fails for the plane (Corollary 3.10), (‡) holds for the surface $Z = \frac{1}{1+X^2+Y^2}$ in $\mathbb{R}^3$.

4.6 Corollary. *For a non-zero linear functional L on $\mathbb{R}[X]$, the following are equivalent:*

(1) *L comes from a positive Borel measure on $\mathbb{R}^n$ supported by K_S.*
(2) *L extends to a linear functional on L' on $\mathbb{R}[X,Y]$ satisfying $L'(T_{S'}) \geq 0$.*

Proof. (1) $\Rightarrow$ (2). Define

$$L'(f) = \int_{\mathbb{R}^n} f(x, \frac{1}{p(x)}) d\mu(x)$$

where μ is the measure. Since $p \geq 1$ on K_S, L' is well-defined. (2) $\Rightarrow$ (1). If $f \in T_S^{\mathrm{alg}}$ then $f \in T_{S'}^{\mathrm{alg}} = T_{S'}^{\mathrm{lin}}$, so $L(f) = L'(f) \geq 0$. (1) follows now, by Haviland's result. $\square$

Remark: T'_S and $T_{S'}$ are related by $T'_S = T_{S'} \cap \mathbb{R}[X]$.

4.7 Example. We use Corollary 4.5 to construct examples where $\dim(K_S) = 1$, K_S is not compact, (‡) holds and T_S is not closed.

(1) Take $n = 1$, $S = \{X^3\}$, $p = 1 + X^2$. Thus $K_{S'}$ is defined by $Y = 1/(1+X^2)$, $X \geq 0$. We claim $T_{S'}$ is not closed. Otherwise, by Corollary 4.5 and Proposition 3.1, $T_{S'}^{\mathrm{alg}} = T_{S'}$ so, intersecting with $\mathbb{R}[X]$, $T_S^{\mathrm{alg}} = T'_S$. Since $X \geq 0$ on K_S, this implies $(1+X^2)^\ell X \in T_S$ for some $\ell \geq 0$, i.e, $(1+X^2)^\ell X = \sigma_0 + \sigma_1 X^3$, $\sigma_i \in \sum \mathbb{R}[X]^2$, $i = 0,1$. Evaluating at $X = 0$, we see that X^2 divides σ_0. Consequently, X divides $(1+X^2)^\ell$, a contradiction.

(2) Take $n = 2$, $S = \{Y^2 - X^3, -(Y^2 - X^3)\}$. Since $Y^2 = X^3$ on K_S, we can take $p = 1 + X^2$. Thus $K_{S'}$ is defined by $Z = 1/(1+X^2)$, $Y^2 = X^3$. We claim that $T_{S'}$ is not closed. Otherwise, as in (1), $(1+X^2)^\ell X \in T_S$ for some $\ell \geq 0$ so, by the method of Example 3.7, $(1+X^2)^\ell X = \sigma_0 + \sigma_1 X^3$ for some $\sigma_i \in \sum \mathbb{R}[X]^2$, $i = 0,1$.

5. Cylinders with compact cross-section

We denote by $\mathbb{R}[X,Y]$ the polynomial ring in $n+1$ variables $X_1, \ldots, X_n, Y$. In this section we consider a subset $S = \{g_1, \ldots, g_s\}$ of $\mathbb{R}[X,Y]$ where the polynomials $g_1, \ldots, g_s$ involve only the variables $X_1, \ldots, X_n$, so K_S has the form $K \times \mathbb{R}$, $K \subseteq \mathbb{R}^n$. We further assume that K is compact. We describe this situation by saying that K_S is a cylinder with compact cross-section. We aim to prove:

5.1 Theorem. *If K_S is a cylinder with compact cross-section, then (‡) holds.*

Proof. Suppose $f \in \mathbb{R}[X,Y]$ is such that $f \geq 0$ on K_S. Fix $d \geq 1$ so that the degree of f as a polynomial in Y is $\leq 2d$. Let $\epsilon > 0$ be given. Let $f_1 = f + \epsilon + \epsilon Y^{2d}$, say

$f_1 = a_0 + a_1 Y + \cdots + a_{2d} Y^{2d}$, $a_i \in \mathbb{R}[X]$. f_1 is strictly positive on $K \times \mathbb{R}$ and its leading coefficient is strictly positive on K. Denote by A the ring of all continuous functions from K to $\mathbb{R}$ and consider $\phi = \tilde{a}_0 + \tilde{a}_1 Y + \cdots + \tilde{a}_{2d} Y^{2d} \in A[Y]$ where $\tilde{a}_i$ is the continuous function corresponding to a_i. Since the roots of ϕ depend continuously on the coefficients, ϕ factors as $\phi = \tilde{a}_{2d}\psi\overline{\psi}$ where the coefficients of ψ are continuous complex-valued functions on K and "bar" denotes complex conjugation. (At each point of K, take $\psi = \prod_{i=1}^{d}(Y - z_i)$ where the $z_1, \ldots, z_d$ are the roots with positive imaginary part, counting multiplicities.) Decomposing ψ as $\psi_1 + i\psi_2$, $\psi_1, \psi_2 \in A[Y]$ this yields

$$\phi = \tilde{a}_{2d}(\psi_1^2 + \psi_2^2). \tag{1}$$

Now use the compactness of K to approximate the coefficients of ψ_1, ψ_2 closely by polynomials in $\mathbb{R}[X]$. In this way we obtain

$$f + \epsilon + \epsilon Y^{2d} = a_{2d}(h_1^2 + h_2^2) + \sum_{i=0}^{2d} b_i Y^i, \tag{2}$$

where h_1, h_2 are polynomials in X, Y and $b_0, \ldots, b_{2d}$ are polynomials in X with $|b_i| < \epsilon$ on K, $i = 0, \ldots, 2d$. By Schmüdgen's Theorem, $a_{2d} \in T$ and $\epsilon \pm b_i \in T$ where T is the preordering in $\mathbb{R}[X]$ generated by $g_1, \ldots, g_s$. This yields

$$\epsilon Y^i + b_i Y^i \in T_S, \text{ for } i \text{ even.} \tag{3}$$

For i odd, say $i = 2k+1$, use the identity $Y^{2k+1} = \frac{1}{2}Y^{2k}((Y+1)^2 - Y^2 - 1)$ plus the fact that $\epsilon Y^{2k}(Y+1)^2 + b_i Y^{2k}(Y+1)^2$, $\epsilon Y^{2k}Y^2 - b_i Y^{2k}Y^2$ and $\epsilon Y^{2k} - b_i Y^{2k}$ all belong to T_S to obtain

$$\epsilon(Y^{i+1} + Y^i + Y^{i-1}) + b_i Y^i \in T_S, \text{ for } i \text{ odd.} \tag{4}$$

Thus, adding $\epsilon(2 + Y + 3Y^2 + Y^3 + 3Y^4 + \cdots + 2Y^{2d})$ to each side of (2), we see finally that

$$f + \epsilon(3 + Y + 3Y^2 + Y^3 + \cdots + 3Y^{2d}) \in T_S. \tag{5}$$

Since the polynomial $q = 3 + Y + 3Y^2 + Y^3 + \cdots + 3Y^{2d}$ depends only on d (not on ϵ), the proof is complete. $\square$

Cylinders with compact cross-section also provide examples where (‡) holds but (†) does not hold:

5.2 Example. Take $n = 2$. Take $S \subseteq \mathbb{R}[X,Y]$ defined by $S = \{X^3(1-X)^3\}$. Then K_S is the strip $[0,1] \times \mathbb{R}$ and (‡) holds by Theorem 5.1. Consider $f = XY^2$. Clearly $f \geq 0$ on K_S. We claim that $\forall \epsilon > 0$, $f + \epsilon \notin T_S$. This will show, in particular, that (†) does not hold. For suppose

$$f + \epsilon = \sigma_0 + \sigma_1 X^3(1-X)^3, \ \sigma_0, \sigma_1 \in \sum \mathbb{R}[X,Y]^2.$$

Comparing degrees in Y, we see that σ_0, σ_1 have the form

$$\sigma_0 = \sum (a_i + b_i Y)^2, \ \sigma_1 = \sum (c_j + d_j Y)^2,$$

with $a_i, b_i, c_j, d_j \in \mathbb{R}[X]$. Equating coefficients of Y^2, this gives

$$X = \sum b_i^2 + \sum d_j^2 X^3(1-X)^3.$$

Evaluating at $X = 0$, we see that X divides each b_i. Dividing through by X, we see that X divides 1, a contradiction.

5.3 Notes.

(1) Example 5.2 generalizes to arbitrary dimension, taking $f = pY^2$ with $p \in \mathbb{R}[X]$, $p \geq 0$ on K and $p \notin T$. Such a polynomial p will exist (regardless of the description of K) if $\dim(K)$ is sufficiently large; see Theorem 3.8.

(2) In Example 5.2 we are not using the "natural" description of the strip $0 \leq X \leq 1$. The question of whether or not (†) holds for the natural description (i.e. $S = \{X, 1-X\}$) is open.

Theorem 5.1 also extends to sets of the form $K \times [0, \infty)$ with K compact:

5.4 Corollary. *Suppose $S' = S \cup \{Y\}$, S as in Theorem 5.1 (so $K_{S'} = K \times [0,\infty)$). Then (‡) holds for S'.*

Note: The result is false if we take the "wrong" description of $K \times [0,\infty)$, e.g., for $S' = S \cup \{Y^3\}$ (∗) fails. (Just modify the proof of Example 3.11.)

Proof. Suppose $f = f(X,Y)$ is non-negative on $K_{S'}$. Then $f(X,Y^2)$ is non-negative on K_S. By the proof of Theorem 5.1, for $q = 3 + Y + 3Y^2 + \cdots + 3Y^{2d}$, d the degree of f in Y, we have, for all real $\epsilon > 0$,

$$f(X,Y^2) + \epsilon q = \sum_e \sigma_e g^e, \tag{1}$$

with $\sigma_e \in \sum \mathbb{R}[X,Y]^2$. Replacing Y by $-Y$, adding, and dividing by 2, using the standard identity

$$\frac{1}{2}(g(Y)^2 + g(-Y)^2) = h(Y^2)^2 + k(Y^2)^2 Y^2 \tag{2}$$

(where, if $g(Y) = \sum_i a_i Y^i$, then $h(Y^2) = \sum_j a_{2j} Y^{2j}$, $k(Y^2) = \sum_j a_{2j+1} Y^{2j}$), this yields

$$f(X,Y^2) + \epsilon q' = \sum_e (\gamma_e + \delta_e Y^2) g^e \tag{3}$$

where $q' = 3 + 3Y^2 + \cdots + 3Y^{2d}$ and $\gamma_e, \delta_e \in \sum \mathbb{R}[X,Y^2]^2$. Replacing Y^2 by Y gives the desired conclusion. $\square$

6. Open problems

1. Does Corollary 5.4 generalize to sets of the form $K \times L$, K compact, L closed semi-algebraic in $\mathbb{R}$ (with the natural description)? We know this is true for L of the form $(-\infty,\infty)$, $[a,\infty)$ or $(-\infty,a]$. Obviously, it is also true for L compact, regardless of the description.

2. Consider the generalized polyhedron K_S in $\mathbb{R}^n$ defined by $S = \{\ell_1,\dots,\ell_s\}$ where $\ell_1,\dots,\ell_s$ are linear. If K_S is compact, we know that (†) holds. (In fact, even ($\dagger_M$) holds.) What if K_S is not compact? If K_S contains a cone of dimension 2 then, by Corollary 3.10, ($*$) fails. This will always be the case if $s < n$. If $s \geq n$ then it may happen that K_S does not contain a cone of dimension 2. Is it true in this case that (‡) holds? We know this is true if $s = n$ (resp., $s = n+1$) by Theorem 5.1 (resp., Corollary 5.4).

3. Find examples where ($*$) holds but (‡) does not hold. Find examples where $T_S^{\text{lin}} \neq \cup_{d\geq 0} \overline{T}_d$. Could it be true that ($*$) $\Rightarrow$ (‡)? This seems highly unlikely.

4. Find a smooth irreducible curve C in $\mathbb{R}^n$ with $\sum \mathbb{R}[C]^2$ not closed in $\mathbb{R}[C]$. Here, $\mathbb{R}[C]$ denotes the coordinate ring of C. According to the "Added in proof" at the end of [9], any such C is necessarily not compact. Note: Our Example 4.7(2) is not smooth.

5. It should be possible to use the method of Corollary 4.5 to construct a curve where (‡) holds but (†) fails. We were not able to do this. In general, it would be nice to know more about the curve case.

6. Find examples where K_S is not compact, $\dim(K_S) \geq 2$, and (†) holds. So far, we only know examples where (‡) holds. Does (†) hold for the strip $0 \leq X \leq 1$ in $\mathbb{R}^2$ with the presentation $S = \{X, 1-X\}$?

7. For $n = 1$ and K_S compact, whether T_S is closed or not depends on the presentation. For example, for $S = \{1 + X, 1 - X\}$, T_S is closed, but for $S = \{(1 - X^2)^3\}$, T_S is not closed. In both cases, $K_S = [-1, 1]$. If $K_S = [-1, 1]^n$, $n \geq 3$, then T_S is never closed, regardless of the presentation (Theorem 3.8). What if $n = 2$? We know T_S is not closed for the presentation $S = \{(1-X^2)^3, (1-Y^2)^3\}$. What about the presentation $S = \{1 + X, 1 - X, 1 + Y, 1 - Y\}$?

8. Certain results are independent of the presentation, i.e., depend only on K_S: If K_S is compact then (†) holds, regardless of the presentation; if $n \geq 2$ and K_S contains a cone of dimension 2 then $(*)$ fails, regardless of the presentation; if K_S contains a cone of dimension n then T_S is closed, regardless of the presentation. Other results depend on the particular presentation. In general, one would like to know to what extent results are independent of the presentation.

References

1. C. Berg, J. Christensen, C. Jensen, *A remark on the multidimensional moment problem*, Math Ann 243 (1979), 163–169.
2. C. Berg, J. Christensen, P. Ressel, *Harmonic analysis on semigroups: theory of positive definite and related functions*, Springer-Verlag, 1984.
3. C. Berg, P.H. Maserick, *Polynomially positive definite sequences*, Math. Ann. 259 (1982), 487–495.
4. E.K. Haviland, *On the momentum problem for distribution functions in more than one dimension*, Amer. J. Math. 57 (1935), 562–572.
5. E.K. Haviland, *On the momentum problem for distribution functions in more than one dimension II*, Amer. J. Math. 58 (1936), 164–168.
6. T. Jacobi, A. Prestel, *Distinguished representations of strictly positive polynomials*, to appear, J. reine angew. Math.
7. M. Marshall, *Extending the archimedean Positivstellensatz to the non-compact case*, to appear, Canad. J. Math.
8. M. Marshall, *Positive polynomials and sums of squares*, Dottorato de Ricerca in Matematica, Dept. di Mat., Univ. Pisa, 2000.
9. C. Scheiderer, *Sums of squares of regular functions on real algebraic varieties*, Transactions AMS 352 (1999), 1030–1069.
10. K. Schmüdgen, *The K-moment problem for compact semi-algebraic sets*, Math. Ann. 289 (1991), 203–206.
11. T. Wörmann, *Short algebraic proofs of theorems of Schmüdgen and Pólya*, preprint.

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