# DISCRETE FOURIER MULTIPLIERS AND CYLINDRICAL BOUNDARY VALUE PROBLEMS

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Abstract. We consider operator-valued boundary value problems in $(0,2\pi)^n$ with periodic or, more generally, $\nu$-periodic boundary conditions. Using the concept of discrete vector-valued Fourier multipliers, we give equivalent conditions for the unique solvability of the boundary value problem. As an application, we study vector-valued parabolic initial boundary value problems in cylindrical domains $(0,2\pi)^n \times V$ with $\nu$-periodic boundary conditions in the cylindrical directions. We show that under suitable assumptions on the coefficients, we obtain maximal $L^q$-regularity for such problems. For symmetric operators such as the Laplacian related results for mixed Dirichlet-Neumann boundary conditions on $(0,2\pi)^n \times V$ are deduced.

## 1. Introduction

In this paper we first study boundary value problems with operator-valued coefficients of the form

$$P(D)u + Q(D)Au = f \quad \text{in } (0,2\pi)^n, \tag{1.1}$$

$$D^\beta u\big|_{x_j=2\pi} - e^{2\pi\nu_j} D^\beta u\big|_{x_j=0} = 0 \quad (j=1,\dots,n,\ |\beta|<m_1). \tag{1.2}$$

Here $P(D)$ is a partial differential operator of order $m_1$ acting on $u=u(x)$ with $x\in(0,2\pi)^n$, $Q(D)$ a partial differential operator of order $m_2 \le m_1$, $A$ is a closed linear operator acting in a Banach space $X$, and $\nu_1,\dots,\nu_n \in \mathbb{C}$ are given numbers. We refer to the boundary conditions as $\nu$-periodic. Note that for $\nu_j = 0$ we have periodic boundary conditions in direction $j$, whereas for $\nu_j = \frac{i}{2}$ we have antiperiodic boundary conditions in this direction. In general, we have different boundary conditions (i.e., different $\nu_j$) in different directions.

As a motivation for studying problem (1.1)-(1.2), we want to mention two classes of problems: First, the boundary value problem (1.1)-(1.2) includes equations of the form

$$u_t(t) + Au(t) = f(t) \quad (t\in(0,2\pi)) \tag{1.3}$$

and

$$u_{tt}(t) - aAu_t(t) - \alpha Au(t) = f(t) \quad (t\in(0,2\pi)) \tag{1.4}$$

with periodic or $\nu$-periodic boundary conditions. Equations of the form (1.3) and (1.4) were considered in [AB02] and [KL06], respectively. These equations fit into

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our context by taking $n = 1$, $P(D) = \partial_t$ and $Q(D) = 1$ for (1.3) and $P(D) = \partial_t^2$, $Q(D) = -a\partial_t - \alpha$ for (1.4).

As a second motivation for studying (1.1)-(1.2), we consider a boundary value problem of cylindrical type where the domain is of the form $\Omega = (0, 2\pi)^n \times V$ with $V \subset \mathbb{R}^{n_V}$ being a sufficiently smooth domain with compact boundary. The operator is assumed to split in the sense that

$$\mathcal{A}(x, D) = P(x^1, D_1) + Q(x^1, D_1)A_V(x^2, D_2) \tag{1.5}$$

where the differential operators $P(x^1, D_1)$ and $Q(x^1, D_1)$ act on $x^1 \in (0, 2\pi)^n$ only and the differential operator $A_V(x^2, D_2)$ acts on $x^2 \in V$ only. The boundary conditions are assumed to be $\nu$-periodic in the $x^1$-direction, whereas in $V$ the operator $A_V(x^2, D_2)$ of order $2m_V$ may be supplemented with general boundary conditions $B_1(x^2, D_2), \ldots, B_{m_V}(x^2, D_2)$. The simplest example of such an operator is the Laplacian in a finite cylinder $(0, 2\pi)^n \times V$ with $\nu$-periodic boundary conditions in the cylindrical directions and Dirichlet boundary conditions on $(0, 2\pi)^n \times \partial V$.

Our first main result (Theorem 3.6) gives, under appropriate assumptions on $P$, $Q$, and $A$, equivalent conditions for the unique solvability of (1.1)-(1.2) in $L^p$-Sobolev spaces. This result generalizes results from [AB02] and [KL06] on equations (1.3) and (1.4), respectively.

In particular in connection with operators of the form (1.5) in cylindrical domains, one is also interested in parabolic theory. Therefore, in Section 5 we study problems of the form

$$\begin{aligned} u_t + \mathcal{A}(x, D)u &= f \quad (t \in [0, T],\, x \in (0, 2\pi)^n \times V), \\ B_j(x, D)u &= 0 \quad (t \in [0, T],\, x \in (0, 2\pi)^n \times \partial V,\, j = 1, \ldots, m_V), \\ (D^\beta u)|_{x_j = 2\pi} - e^{2\pi\nu_j}(D^\beta u)|_{x_j = 0} &= 0 \quad (j = 1, \ldots, n;\ |\beta| < m_1), \\ u(0, x) &= u_0(x) \quad (x \in (0, 2\pi)^n \times V). \end{aligned} \tag{1.6}$$

Here $\mathcal{A}(x, D)$ is defined as in (1.5). If $(A_V, B_1, \ldots, B_{m_V})$ is a parabolic boundary value problem in the sense of parameter-ellipticity (see [DHP03, Section 8]), we obtain, under suitable assumptions on $P$ and $Q$, maximal $L^q$-regularity for (1.6) (see Theorems 4.3 and 4.6 below). The proof of maximal regularity is based on the $\mathcal{R}$-boundedness of the resolvent related to (1.6).

Periodic boundary values appear for instance in the study of the formation of keratin networks, which are a component of the cytoskeleton of biological cells. In [ABF$^+$08] the evolution of a pool of soluble polymers fueling network growth is modeled by the Laplace operator with periodic boundary conditions.

Apart from its own interest, the consideration of $\nu$-periodic boundary conditions also allows us to address boundary conditions of mixed type. As the simplest example, when $a = 0$ we can analyze equation (1.4) with Dirichlet-Neumann type boundary conditions

$$u(0) = 0,\ u_t(\pi) = 0.$$

The connection to periodic and antiperiodic boundary conditions is given by suitable extensions of the solution. This was also considered in [AB02] where – starting from periodic boundary conditions – the pure Dirichlet and the pure Neumann case could be treated.

The main tool to address problems (1.1)-(1.2) and (1.6) is the theory of discrete vector-valued Fourier multipliers. Taking the Fourier series in the cylindrical directions, we are faced with the question under which conditions an operator-valued Fourier series defines a bounded operator in $L^p$. This question was answered by Arendt and Bu in [AB02] for the one-dimensional case $n = 1$, where a discrete operator-valued Fourier multiplier result for UMD spaces and applications to periodic Cauchy problems of first and second order in Lebesgue- and Hölder-spaces can be found. For general $n$, the main result on vector-valued Fourier multipliers is contained in [BK04]. A shorter proof of this result by means of induction based on the result for $n = 1$ in [AB02] is given in [Bu06]. As pointed out by the authors in [AB02] and [BK04], the results can as well be deduced from [ŠW07, Theorems 3.7, 3.8].

A generalization of the results in [AB02] to periodic first order integro-differential equations in Lebesgue-, Besov- and Hölder-spaces is given in [KL04]. Here the concept of 1-regularity in the context of sequences is introduced (see Remark 2.11 below).

In [KL06] one finds a comprehensive treatment of periodic second order differential equations of type (1.4) in Lebesgue- and Hölder-spaces. In particular, the special case of a Cauchy problem of second order, i.e. $\alpha = 0, a = 1$, where $A$ is the generator of a strongly continuous cosine function is investigated. In [KLP09] more general equations are treated in the mentioned spaces as well as in Triebel-Lizorkin-spaces. Moreover, applications to nonlinear equations are presented.

Maximal regularity of second order initial value problems of the type

$$\begin{aligned} u_{tt}(t) + Bu_t(t) + Au(t) &= f(t) \quad (t \in [0,T)), \\ u(0) = u_t(0) &= 0 \end{aligned}$$

is treated in [CS05] and [CS08]. In particular, $p$-independence of maximal regularity for second order problems of this type is shown. The same equation involving dynamic boundary conditions is studied in [XL04]. The non-autonomous second order problem, involving $t$-dependent operators $B(t)$ and $A(t)$, is treated in [BCS08]. We also refer to [XL98] for the treatment of higher order Cauchy problems.

In [AR09] various properties as e.g. Fredholmness of the operator $\partial_t - A(\cdot)$ associated to the non-autonomous periodic first order Cauchy-problem in $L^p$-context are investigated. Results on this operator based on Floquet theory are obtained in the PhD-thesis [Gau01]. We remark that in Floquet theory $\nu$-periodic (instead of periodic) boundary conditions appear in a natural way.

For the treatment of boundary value problems in $(0, 1)$ with operator-valued coefficients subject to numerous types of homogeneous and inhomogeneous boundary conditions, we refer to [FLM$^+$08], [FSY09], [FY10] and the references therein. Their approaches mainly rely on semigroup theory and do not allow for an easy generalization to $(0, 1)^n$. In [FSY09] however, applications to boundary value problems in the cylindrical space domain $(0, 1) \times V$ can be found.

The usage of operator-valued multipliers to treat cylindrical in space boundary value problems was first carried out in [Gui04] and [Gui05] in a Besov-space setting. In these papers the author constructs semiclassical fundamental solutions for a class of elliptic operators on infinite cylindrical domains $\mathbb{R}^n \times V$. This proves to be a strong tool for the treatment of related elliptic and parabolic ([Gui04] and [Gui05]), as well as of hyperbolic ([Gui05]) problems. Operators in cylindrical domains with a similar splitting property as in the present paper were, in the case of an infinite cylinder, also considered in [NS].

## 2. Discrete Fourier multipliers and $\mathcal{R}$-boundedness

In the following, let $X$ and $Y$ be Banach spaces, $1 < p < \infty$, $n \in \mathbb{N}$, and $\mathcal{Q}_n := (0, 2\pi)^n$. By $\mathcal{L}(X, Y)$ we denote the space of all bounded linear operators from $X$ to $Y$, and we set $\mathcal{L}(X) := \mathcal{L}(X, X)$. By $L^p(\mathcal{Q}_n, X)$ we denote the standard Bochner space of $X$-valued $L^p$-functions defined on $\mathcal{Q}_n$. For $f \in L^p(\mathcal{Q}_n, X)$ and $\mathbf{k} \in \mathbb{Z}^n$ the $\mathbf{k}$-th Fourier coefficient of $f$ is given by

$$\hat{f}(\mathbf{k}) := \frac{1}{(2\pi)^n} \int_{\mathcal{Q}_n} e^{-i\mathbf{k}\cdot x} f(x) dx \,. \tag{2.1}$$

By Fejer's Theorem we see that $f(x) = 0$ almost everywhere if $\hat{f}(\mathbf{k}) = 0$ for all $\mathbf{k} \in \mathbb{Z}^n$ as well as $f(x) = \hat{f}(\mathbf{0})$ almost everywhere if $\hat{f}(\mathbf{k}) = 0$ for all $\mathbf{k} \in \mathbb{Z}^n \setminus \{\mathbf{0}\}$. Moreover for $f, g \in L^p(\mathcal{Q}_n, X)$ and a closed operator $A$ in $X$ it holds that $f(x) \in D(A)$ and $Af(x) = g(x)$ almost everywhere if and only if $\hat{f}(\mathbf{k}) \in D(A)$ and $A\hat{f}(\mathbf{k}) = \hat{g}(\mathbf{k})$ for all $\mathbf{k} \in \mathbb{Z}^n$. We will frequently make use of these observations without further comments.

*Definition* 2.1. A function $M\colon \mathbb{Z}^n \to \mathcal{L}(X, Y)$ is called a (discrete) $L^p$-multiplier if for each $f \in L^p(\mathcal{Q}_n, X)$ there exists a $g \in L^p(\mathcal{Q}_n, Y)$ such that

$$\hat{g}(\mathbf{k}) = M(\mathbf{k})\hat{f}(\mathbf{k}) \quad (\mathbf{k} \in \mathbb{Z}^n).$$

In this case there exists a unique operator $T_M \in \mathcal{L}(L^p(\mathcal{Q}_n, X), L^p(\mathcal{Q}_n, Y))$ associated to $M$ such that

$$(T_M f)\hat{\ }(\mathbf{k}) = M(\mathbf{k})\hat{f}(\mathbf{k}) \quad (\mathbf{k} \in \mathbb{Z}^n). \tag{2.2}$$

The property of being a Fourier multiplier is closely related to the concept of $\mathcal{R}$-boundedness. Here we give only the definition and some properties which will be used later on; as references for $\mathcal{R}$-boundedness we mention [KW04] and [DHP03].

*Definition* 2.2. A family $\mathcal{T} \subset \mathcal{L}(X, Y)$ is called $\mathcal{R}$-bounded if there exist a $C > 0$ and a $p \in [1, \infty)$ such that for all $N \in \mathbb{N}$, $T_j \in \mathcal{T}$, $x_j \in X$ and all independent symmetric $\{-1, 1\}$-valued random variables $\varepsilon_j$ on a probability space $(\Omega, \mathcal{A}, P)$ for $j = 1, ..., N$, we have that

$$\Big\| \sum_{j=1}^N \varepsilon_j T_j x_j \Big\|_{L^p(\Omega, Y)} \le C_p \Big\| \sum_{j=1}^N \varepsilon_j x_j \Big\|_{L^p(\Omega, X)}. \tag{2.3}$$

The smallest $C_p > 0$ such that (2.3) is satisfied is called $\mathcal{R}_p$-bound of $\mathcal{T}$ and denoted by $\mathcal{R}_p(\mathcal{T})$.

By Kahane's inequality, (2.3) holds for all $p \in [1, \infty)$ if it holds for one $p \in [1, \infty)$. Therefore, we will drop the $p$-dependence of $\mathcal{R}_p(\mathcal{T})$ in the notation and write $\mathcal{R}(\mathcal{T})$.

**Lemma 2.3.** *a) Let $Z$ be a third Banach space and let $\mathcal{T}, \mathcal{S} \subset \mathcal{L}(X,Y)$ as well as $\mathcal{U} \subset \mathcal{L}(Y,Z)$ be $\mathcal{R}$-bounded. Then $\mathcal{T}+\mathcal{S}$, $\mathcal{T}\cup\mathcal{S}$ and $\mathcal{U}\mathcal{T}$ are $\mathcal{R}$-bounded as well and we have*

$$\mathcal{R}(\mathcal{T}+\mathcal{S}),\ \mathcal{R}(\mathcal{T}\cup\mathcal{S}) \le \mathcal{R}(\mathcal{S})+\mathcal{R}(\mathcal{T}), \quad \mathcal{R}(\mathcal{U}\mathcal{T}) \le \mathcal{R}(\mathcal{U})\mathcal{R}(\mathcal{T}).$$

*Furthermore, if $\overline{\mathcal{T}}$ denotes the closure of $\mathcal{T}$ with respect to the strong operator topology, then we have $\mathcal{R}(\overline{\mathcal{T}}) = \mathcal{R}(\mathcal{T})$.*

*b)* Contraction principle of Kahane: *Let $p \in [1,\infty)$. Then for all $N \in \mathbb{N}, x_j \in X, \varepsilon_j$ as above, and for all $a_j, b_j \in \mathbb{C}$ with $|a_j| \le |b_j|$ for $j = 1,\dots,N$ we have*

$$\Big\| \sum_{j=1}^N a_j\varepsilon_j x_j \Big\|_{L^p(\Omega,X)} \le 2 \Big\| \sum_{j=1}^N b_j\varepsilon_j x_j \Big\|_{L^p(\Omega,X)}. \tag{2.4}$$

For $M\colon \mathbb{Z}^n \to \mathcal{L}(X,Y)$ and $1 \le j \le n$ we inductively define the differences (discrete derivatives)

$$\Delta_j^\ell M(\mathbf{k}) := \Delta_j^{\ell-1} M(\mathbf{k}) - \Delta_j^{\ell-1} M(\mathbf{k}-\mathbf{e}_j) \quad (\ell \in \mathbb{N},\, \mathbf{k} \in \mathbb{Z}^n),$$

where $\mathbf{e}_j$ denotes the $j$-th unit vector in $\mathbb{R}^n$ and where we have set $\Delta_j^0 M(\mathbf{k}) := M(\mathbf{k})$ ($\mathbf{k} \in \mathbb{Z}^n$). As $\Delta_i^{\gamma_i}$ and $\Delta_j^{\gamma_j}$ commute for $1 \le i,j \le n$, for a multi-index $\gamma \in \mathbb{N}_0^n$ the expression

$$\Delta^\gamma M(\mathbf{k}) := \big(\Delta_1^{\gamma_1} \cdots \Delta_n^{\gamma_n} M\big)(\mathbf{k}) \quad (\mathbf{k} \in \mathbb{Z}^n)$$

is well-defined. Given $\alpha, \beta, \gamma \in \mathbb{N}_0^n$, we will write $\alpha \le \gamma \le \beta$ if $\alpha_j \le \gamma_j \le \beta_j$ for all $1 \le j \le n$. We also set $|\alpha| := \alpha_1 + \cdots + \alpha_n$, $\mathbf{0} := (0,\dots,0)$, and $\mathbf{1} := (1,\dots,1)$. We agree to write $\mathbf{0} < \gamma$ if $\mathbf{0} \le \gamma$ and $0 < \gamma_j$ for at least one $1 \le j \le n$.

We recall that a Banach space $X$ is called a UMD space or a Banach space of class $\mathcal{HT}$ if there exists a $q \in (1,\infty)$ (equivalently: if for all $q \in (1,\infty)$) the Hilbert transform defines a bounded operator in $L^q(\mathbb{R},X)$. A Banach space $X$ is said to have property $(\alpha)$ if there exists a $C > 0$ such that for all $N \in \mathbb{N}$, $\alpha_{ij} \in \mathbb{C}$ with $|\alpha_{ij}| \le 1$, all $x_{ij} \in X$, and all independent symmetric $\{+1,-1\}$-valued random variables $\varepsilon_i^{(1)}$ on a probability space $(\Omega_1, \mathcal{A}_1, P_1)$ and $\varepsilon_j^{(2)}$ on a probability space $(\Omega_2, \mathcal{A}_2, P_2)$ for $i,j = 1,\dots,N$ we have

$$\Big\| \sum_{i,j=1}^N \alpha_{ij}\varepsilon_i^{(1)}\varepsilon_j^{(2)} x_{ij} \Big\|_{L^2(\Omega_1\times\Omega_2,X)} \le C \Big\| \sum_{i,j=1}^N \varepsilon_i^{(1)}\varepsilon_j^{(2)} x_{ij} \Big\|_{L^2(\Omega_1\times\Omega_2,X)}.$$

The following result from Bu and Kim characterizes discrete Fourier multipliers by $\mathcal{R}$-boundedness.

**Theorem 2.4** ([BK04]). *a) Let $X,Y$ be UMD spaces and let $\mathcal{T} \subset \mathcal{L}(X,Y)$ be $\mathcal{R}$-bounded. If $M\colon \mathbb{Z}^n \to \mathcal{L}(X,Y)$ satisfies*

$$\big\{|\mathbf{k}|^{|\gamma|}\Delta^\gamma M(\mathbf{k}) :\ \mathbf{k} \in \mathbb{Z}^n \setminus [-1,1]^n,\ \mathbf{0} < \gamma \le \mathbf{1}\big\} \cup \big\{M(\mathbf{k}) :\ \mathbf{k} \in \mathbb{Z}^n\big\} \subset \mathcal{T}, \tag{2.5}$$

*then $M$ defines a Fourier multiplier.*

*b) If $X,Y$ additionally enjoy property $(\alpha)$, then*

$$\big\{\mathbf{k}^\gamma\Delta^\gamma M(\mathbf{k}) :\ \mathbf{k} \in \mathbb{Z}^n \setminus [-1,1]^n,\ \mathbf{0} < \gamma \le \mathbf{1}\big\} \cup \big\{M(\mathbf{k}) :\ \mathbf{k} \in \mathbb{Z}^n\big\} \subset \mathcal{T} \tag{2.6}$$

*is sufficient. In this case the set*

$$\{T_M :\ M \text{ satisfies condition } (2.6)\} \subset \mathcal{L}\big(L^p(\mathcal{Q}_n,X), L^p(\mathcal{Q}_n,Y)\big)$$

*is $\mathcal{R}$-bounded again.*

*Remark* 2.5. In [BK04], Theorem 2.4 is stated with discrete derivatives $\tilde{\Delta}$ defined in such a way that $\Delta^\gamma M(\mathbf{k}+\gamma) = \tilde{\Delta}^\gamma M(\mathbf{k})$. However, as for fixed $\mathbf{0} \le \gamma \le \mathbf{1}$ there exist $c, C > 0$ such that $c|\mathbf{k}-\gamma| \le |\mathbf{k}| \le C|\mathbf{k}-\gamma|$ for $\mathbf{k} \in \mathbb{Z}^n \setminus [-1,1]^n$, Lemma 2.3 shows our formulation to be equivalent to the one in [BK04]. Throughout this article, we will make use of this estimate frequently without any further comment. Furthermore, the authors in [BK04] chose the slightly stronger conditions

$$\big\{|\mathbf{k}|^{|\gamma|}\Delta^\gamma M(\mathbf{k}) :\ \mathbf{k} \in \mathbb{Z}^n,\ \mathbf{0} \le \gamma \le \mathbf{1}\big\} \subset \mathcal{T} \tag{2.7}$$

and

$$\big\{\mathbf{k}^{\gamma}\Delta^\gamma M(\mathbf{k}) :\ \mathbf{k} \in \mathbb{Z}^n,\ \mathbf{0} \le \gamma \le \mathbf{1}\big\} \subset \mathcal{T} \tag{2.8}$$

in their article. However, the proof is the same and conditions (2.5) and (2.6) are more convenient to verify.

The following lemma states some properties for discrete derivatives, where $(S_{\mathbf{k}})_{\mathbf{k}\in\mathbb{Z}^n}$ and $(T_{\mathbf{k}})_{\mathbf{k}\in\mathbb{Z}^n}$ denote arbitrary commuting sequences in $\mathcal{L}(X)$. For $\alpha \in \mathbb{N}_0^n \setminus \{\mathbf{0}\}$, let

$$\mathcal{Z}_\alpha := \Big\{\mathcal{W} = (\omega^1, \dots, \omega^r);\ 1 \le r \le |\alpha|,\ \mathbf{0} \le \omega^j \le \alpha,\, \omega^j \ne \mathbf{0},\, \sum_{j=1}^r \omega^j = \alpha\Big\}$$

denote the set of all additive decompositions of $\alpha$ into $r = r_{\mathcal{W}}$ multi-indices and set $\mathcal{Z}_{\mathbf{0}} := \{\emptyset\}$ and $r_\emptyset := 0$. For $\mathcal{W} \in \mathcal{Z}_\alpha$ we set $\omega_j^* := \sum_{l=j+1}^r \omega^l$. In the following, $c_{\alpha,\beta}$ and $c_{\mathcal{W}}$ will denote integer constants depending on $\alpha, \beta$ and $\mathcal{W}$, respectively.

**Lemma 2.6.** *a)* Leibniz rule: *For $\alpha \in \mathbb{N}_0^n$ and $\mathbf{k} \in \mathbb{Z}^n$ we have*

$$\Delta^\alpha\big(ST\big)(\mathbf{k}) = \sum_{\mathbf{0}\le\beta\le\alpha} c_{\alpha,\beta}\big(\Delta^{\alpha-\beta}S\big)(\mathbf{k}-\beta)\big(\Delta^\beta T\big)(\mathbf{k}).$$

*b) Let $\big(S^{-1}\big)(\mathbf{k}) := \big(S_{\mathbf{k}}\big)^{-1}$ exist for all $\mathbf{k} \in \mathbb{Z}^n$. Then, for $\alpha \in \mathbb{N}_0^n$ and $\mathbf{k} \in \mathbb{Z}^n$ we have*

$$\Delta^\alpha\big(S^{-1}\big)(\mathbf{k}) = \sum_{\mathcal{W}\in\mathcal{Z}_\alpha} c_{\mathcal{W}}\big(S^{-1}\big)(\mathbf{k}-\alpha)\prod_{j=1}^{r_{\mathcal{W}}}\Big(\big(\Delta^{\omega^j}S\big)S^{-1}\Big)(\mathbf{k}-\omega_j^*).$$

*Proof.* We will show both assertions by induction on $|\alpha|$, the case $|\alpha| = 0$ being obvious.

a) By definition, we have

$$\big(\Delta^{\mathbf{e}_j}(ST)\big)(\mathbf{k}) = \big(ST\big)(\mathbf{k}) - \big(ST\big)(\mathbf{k}-\mathbf{e}_j) = S(\mathbf{k}-\mathbf{e}_j)\big(\Delta^{\mathbf{e}_j}T\big)(\mathbf{k}) + \big(\Delta^{\mathbf{e}_j}S\big)(\mathbf{k})T(\mathbf{k}),$$

and for $\alpha' := \alpha - \mathbf{e}_j$ where $\alpha_j \ne 0$ we obtain

$$\begin{aligned}\big(\Delta^\alpha(ST)\big)(\mathbf{k}) &= \Delta^{\mathbf{e}_j}\sum_{\beta\le\alpha'} c_{\alpha'\beta}\big(\Delta^{\alpha'-\beta}S\big)(\mathbf{k}-\beta)\big(\Delta^\beta T\big)(\mathbf{k}) \\ &= \sum_{\beta\le\alpha} c_{\alpha\beta}\big(\Delta^{\alpha-\beta}S\big)(\mathbf{k}-\beta)\big(\Delta^\beta T\big)(\mathbf{k}).\end{aligned}$$

b) For $|\alpha| \geq 1$, we apply a) to $SS^{-1}$ and get

$$\begin{aligned} 0 &= \big(\Delta^\alpha (SS^{-1})\big)(\mathbf{k}) \\ &= S(\mathbf{k}-\alpha)\big(\Delta^\alpha S^{-1}\big)(\mathbf{k}) + \sum_{\beta<\alpha} c_{\alpha\beta}\big(\Delta^{\alpha-\beta} S\big)(\mathbf{k}-\beta)\big(\Delta^\beta S^{-1}\big)(\mathbf{k}). \end{aligned}$$

Hence

$$\begin{aligned} &\big(\Delta^\alpha S^{-1}\big)(\mathbf{k}) = -S^{-1}(\mathbf{k}-\alpha)\sum_{\beta<\alpha} c_{\alpha\beta}\big(\Delta^{\alpha-\beta} S\big)(\mathbf{k}-\beta)\big(\Delta^\beta S^{-1}\big)(\mathbf{k}) \\ &\quad = -\sum_{\beta<\alpha}\sum_{\mathcal{W}\in\mathcal{Z}_\beta} c_{\mathcal{W}} S^{-1}(\mathbf{k}-\alpha)\Big(\big(\Delta^{\alpha-\beta}S\big)S^{-1}\Big)(\mathbf{k}-\beta)\prod_{j=1}^{r_{\mathcal{W}}}\Big(\big(\Delta^{\omega^j}S\big)S^{-1}\Big)(\mathbf{k}-\omega_j^*) \\ &\quad = \sum_{\mathcal{W}\in\mathcal{Z}_\alpha} c_{\mathcal{W}} S^{-1}(\mathbf{k}-\alpha)\Big(\big(\Delta^{\omega^1}S\big)S^{-1}\Big)(\mathbf{k}-\omega_1^*)\prod_{j=2}^{r_{\mathcal{W}}}\Big(\big(\Delta^{\omega^j}S\big)S^{-1}\Big)(\mathbf{k}-\omega_j^*). \end{aligned}$$

□

*Definition* 2.7. Consider a polynomial $P : \mathbb{R}^n \to \mathbb{C};\ \xi \mapsto P(\xi)$ and let $P^\#$ denote its principal part.

a) $P$ is called elliptic if $P^\#(\xi) \neq 0$ for $\xi \in \mathbb{R}^n \setminus \{\mathbf{0}\}$.

b) Let $\phi \in (0,\pi)$ and let $\Sigma_\phi := \{\lambda \in \mathbb{C}\setminus\{0\} : |\arg(\lambda)| < \phi\}$ be the open sector with angle $\phi$. Then $P$ is called parameter-elliptic in $\overline{\Sigma}_{\pi-\phi}$ if $\lambda + P^\#(\xi) \neq 0$ for $(\lambda,\xi) \in \overline{\Sigma}_{\pi-\phi}\times\mathbb{R}^n\setminus\{(0,\mathbf{0})\}$. In this case,

$$\varphi_P := \inf\{\phi \in (0,\pi) : P \text{ is parameter-elliptic in } \overline{\Sigma}_{\pi-\phi}\}$$

is called the angle of parameter-ellipticity of $P$.

*Remark* 2.8. a) By quasi-homogeneity of $(\lambda,\xi)\mapsto \lambda + P^\#(\xi)$, we easily see that $P$ is parameter-elliptic in $\overline{\Sigma}_{\pi-\phi}$ if and only if for all polynomials $N$ with $\deg N \leq \deg P$ there exist $C > 0$ and a bounded subset $G \subset \mathbb{R}^n$ such that the estimate $|\xi|^m|N(\xi)| \leq C|\lambda + P(\xi)|$ holds for all $\lambda \in \overline{\Sigma}_{\pi-\phi}$, all $0 \leq m \leq \deg P - \deg N$ and all $\xi \in \mathbb{R}^n \setminus G$.

b) In the same way, $P$ is elliptic if and only if the assertion in a) is valid for $\lambda = 0$.

c) By induction, one can see that for $|\alpha| \leq \deg P$ the discrete polynomial $\Delta^\alpha P(\mathbf{k})$ defines a polynomial of degree not greater than $\deg P - |\alpha|$. If $P$ is elliptic, this implies $|\mathbf{k}|^{|\alpha|}|\Delta^\alpha P(\mathbf{k})| \leq C|P(\mathbf{k})|$ ($\mathbf{k} \in \mathbb{Z}^n \setminus G$) with a finite set $G \subset \mathbb{Z}^n$.

In what follows the assumption that $(\lambda + \mu A)^{-1}$ exists for $\lambda, \mu \in \mathbb{C}$ is meant to imply both $(\lambda+\mu A)^{-1} \in \mathcal{L}(X)$ and $(\lambda+\mu A)^{-1}(X) = D(A)$. Hence $\mu \neq 0$ and $\lambda \in \rho(-\mu A)$.

**Proposition 2.9.** *Let $A$ be a closed linear operator in a UMD space $X$. Consider polynomials $P, Q : \mathbb{Z}^n \to \mathbb{C}$ such that*

- *$P$ and $Q$ are elliptic,*
- *$\big(P(\mathbf{k}) + Q(\mathbf{k})A\big)^{-1}$ exists for all $\mathbf{k} \in \mathbb{Z}^n$,*
- *$\Big\{P(\mathbf{k})\big(P(\mathbf{k}) + Q(\mathbf{k})A\big)^{-1} : \mathbf{k} \in \mathbb{Z}^n\Big\}$ is $\mathcal{R}$-bounded.*

*Then for every polynomial $N$ with* $\deg N \leq \deg P$ *the map*

$$M : \mathbb{Z}^n \to \mathcal{L}(X) : \quad \mathbf{k} \mapsto N(\mathbf{k})\big(P(\mathbf{k}) + Q(\mathbf{k})A\big)^{-1}$$

*defines an $L^p$-multiplier for $1 < p < \infty$.*

*Proof.* Lemma 2.6 yields

$$|\mathbf{k}|^{|\gamma|}\Delta^\gamma M(\mathbf{k}) = \sum_{\beta\leq\gamma}\sum_{\mathcal{W}\in\mathcal{Z}_\beta} c_{\mathcal{W}}|\mathbf{k}|^{|\gamma-\beta|}(\Delta^{\gamma-\beta}N)(\mathbf{k}-\beta)\big(P(\mathbf{k}-\beta)+Q(\mathbf{k}-\beta)A\big)^{-1}$$
$$\cdot\prod_{j=1}^{r_{\mathcal{W}}}|\mathbf{k}|^{|\omega^j|}\big(\Delta^{\omega^j}P(\mathbf{k}-\omega_j^*)+\Delta^{\omega^j}Q(\mathbf{k}-\omega_j^*)A\big)\big(P(\mathbf{k}-\omega_j^*)+Q(\mathbf{k}-\omega_j^*)A\big)^{-1}.$$

By Remark 2.8, we know that $\deg(\Delta^{\gamma-\beta}N) \leq \deg N - |\gamma-\beta|$. This and the ellipticity of $P$ imply $|\mathbf{k}|^{|\gamma-\beta|}|\Delta^{\gamma-\beta}N(\mathbf{k})| \leq C|P(\mathbf{k})|$ for $\mathbf{k}\in\mathbb{Z}^n\setminus G$ with a finite set $G\subset\mathbb{Z}^n$. By Kahane's contraction principle, we obtain the $\mathcal{R}$-boundedness of

$$\Big\{|\mathbf{k}|^{|\gamma-\beta|}\Delta^{\gamma-\beta}N(\mathbf{k}-\beta)\big(P(\mathbf{k}-\beta)+Q(\mathbf{k}-\beta)A\big)^{-1} : \mathbf{k}\in\mathbb{Z}^n\setminus G\Big\}.$$

Since

$$Q(\mathbf{k})A\big(P(\mathbf{k})+Q(\mathbf{k})A\big)^{-1} = \mathrm{id}_X - P(\mathbf{k})\big(P(\mathbf{k})+Q(\mathbf{k})A\big)^{-1},$$

in the same way the $\mathcal{R}$-boundedness of

$$\Big\{|\mathbf{k}|^{|\omega^j|}\Delta^{\omega^j}Q(\mathbf{k}-\omega_j^*)A\big(P(\mathbf{k}-\omega_j^*)+Q(\mathbf{k}-\omega_j^*)A\big)^{-1} : \mathbf{k}\in\mathbb{Z}^n\setminus G\Big\}$$

follows from the ellipticity of $Q$. Now the assertion follows from Lemma 2.3 and Theorem 2.4. $\square$

Proposition 2.9 is closely related to the concept of 1-regularity of complex-valued sequences, introduced in [KL04] for the one dimensional case $n = 1$. In fact, if $Q(\mathbf{k}) \neq 0$ for all $\mathbf{k}\in\mathbb{Z}^n$, we may write $M(\mathbf{k}) = \frac{N(\mathbf{k})}{Q(\mathbf{k})}\big(\frac{P(\mathbf{k})}{Q(\mathbf{k})}+A\big)^{-1}$. Hence, for $n = 1$ we enter the framework of [KLP09, Proposition 5.3], i.e. $M(k) = a_k(b_k - A)^{-1}$ with $(a_k)_{k\in\mathbb{Z}}, (b_k)_{k\in\mathbb{Z}} \subset \mathbb{C}$. We will give a generalization of this concept to arbitrary $n$ and briefly indicate the connection to the results above.

*Definition* 2.10. We call a pair of sequences $(a_{\mathbf{k}}, b_{\mathbf{k}})_{\mathbf{k}\in\mathbb{Z}^n} \subset \mathbb{C}^2$ 1-regular if for all $\mathbf{0}\leq\gamma\leq\mathbf{1}$ there exist a finite set $K\subset\mathbb{Z}^n$ and a constant $C > 0$ such that

$$|\mathbf{k}^\gamma|\max\{|(\Delta^\gamma a)_{\mathbf{k}}|, |(\Delta^\gamma b)_{\mathbf{k}}|\} \leq C|b_{\mathbf{k}}| \quad (\mathbf{k}\in\mathbb{Z}^n\setminus K). \tag{2.9}$$

We say the pair $(a_{\mathbf{k}}, b_{\mathbf{k}})_{\mathbf{k}\in\mathbb{Z}^n}$ is strictly 1-regular if $|\mathbf{k}^\gamma|$ can be replaced by $|\mathbf{k}|^{|\gamma|}$ in (2.9). A sequence $(a_{\mathbf{k}})_{\mathbf{k}\in\mathbb{Z}^n}$ is called (strictly) 1-regular if $(a_{\mathbf{k}}, a_{\mathbf{k}})_{\mathbf{k}\in\mathbb{Z}^n}$ has this property.

*Remark* 2.11. a) In the case $n = 1$, a sequence $(a_k)_{k\in\mathbb{Z}} \subset \mathbb{C}\setminus\{0\}$ is 1-regular in $\mathbb{Z}$ in the sense of Definition 2.10 if and only if the sequence $\big(\frac{k(a_{k+1}-a_k)}{a_k}\big)_{k\in\mathbb{Z}}$ is bounded. Hence our definition extends the one from [KL04] for a sequence $(a_k)_{k\in\mathbb{Z}}$.

b) With $\gamma = 0$ the definition especially requests $|a_{\mathbf{k}}| \leq C|b_{\mathbf{k}}|$ for $\mathbf{k}\in\mathbb{Z}^n\setminus K$.

c) Strict 1-regularity implies 1-regularity. If $n = 1$ both concepts are equivalent.

d) Subject to the assumptions of Proposition 2.9, let $Q(\mathbf{k}) \neq 0$ for $\mathbf{k}\in\mathbb{Z}^n$. Then the pair $\big(a_{\mathbf{k}}, b_{\mathbf{k}}\big)_{\mathbf{k}\in\mathbb{Z}^n}$ with $a_{\mathbf{k}} := \frac{N(\mathbf{k})}{Q(\mathbf{k})}$, $b_{\mathbf{k}} := \frac{P(\mathbf{k})}{Q(\mathbf{k})}$ is strictly 1-regular.

e) Again from Lemma 2.6 we deduce the following variant of Proposition 2.9: Let $b_{\mathbf{k}} \in \rho(A)$ for all $\mathbf{k} \in \mathbb{Z}^n$, let $\mathcal{R}(\{b_{\mathbf{k}}(b_{\mathbf{k}} - A)^{-1} : \ \mathbf{k} \in \mathbb{Z}^n \setminus G\}) < \infty$ for some finite subset $G \subset \mathbb{Z}^n$, and let $(a_{\mathbf{k}}, b_{\mathbf{k}})_{\mathbf{k}\in\mathbb{Z}^n}$ be strictly 1-regular. Then $M(\mathbf{k}) := a_{\mathbf{k}}(b_{\mathbf{k}} - A)^{-1}$ defines a Fourier multiplier.

## 3. $\nu$-periodic boundary value problems

*Definition* 3.1. Let $X$ be a Banach space, $m \in \mathbb{N}_0$, $n \in \mathbb{N}$ and $\nu \in \mathbb{C}^n$. We set $D^\alpha := D_1^{\alpha_1} \dots D_n^{\alpha_n}$ with $D_j = -i\frac{\partial}{\partial j}$ and denote by $W^{m,p}_{\nu,per}(\mathcal{Q}_n, X)$ the space of all $u \in W^{m,p}(\mathcal{Q}_n, X)$ such that for all $j \in \{1, \dots, n\}$ and all $|\alpha| < m$ it holds that

$$(D^\alpha u)|_{x_j=2\pi} = e^{2\pi\nu_j}(D^\alpha u)|_{x_j=0}.$$

For sake of convenience we set $W^{m,p}_{per}(\mathcal{Q}_n, X) := W^{m,p}_{0,per}(\mathcal{Q}_n, X)$.

We give some helpful characterizations of the space $W^{m,p}_{\nu,per}(\mathcal{Q}_n, X)$ where we omit the rather simple proof.

**Lemma 3.2.** *The following assertions are equivalent:*

(i) $u \in W^{m,p}_{\nu,per}(\mathcal{Q}_n, X)$.

(ii) $u \in W^{m,p}(\mathcal{Q}_n, X)$ *and for all* $|\alpha| \le m$ *it holds that*

$$(e^{-\nu\cdot}D^\alpha u)\hat{}(\mathbf{k}) = (\mathbf{k} - i\nu)^\alpha (e^{-\nu\cdot}u)\hat{}(\mathbf{k})$$

*for all* $\mathbf{k} \in \mathbb{Z}^n$.

(iii) *There exists* $v \in W^{m,p}_{per}(\mathcal{Q}_n, X)$ *such that* $u = e^{\nu\cdot}v$.

The following lemma characterizes multipliers such that the associated operators map $L^p(\mathcal{Q}_n, X)$ into $W^{\alpha,p}_{per}(\mathcal{Q}_n, X)$. The proof follows the one for the case $n = 1$ of [AB02, Lemma 2.2].

**Lemma 3.3.** *Let* $1 \le p < \infty$, $m \in \mathbb{N}$ *and* $M: \mathbb{Z}^n \to \mathcal{L}(X)$. *Then the following assertions are equivalent:*

(i) $M$ *is an* $L^p$*-multiplier such that the associated operator* $T_M \in \mathcal{L}(L^p(\mathcal{Q}_n, X))$ *maps* $L^p(\mathcal{Q}_n, X)$ *into* $W^{m,p}_{per}(\mathcal{Q}_n, X)$.

(ii) $M_\alpha: \mathbb{Z}^n \to \mathcal{L}(X)$, $\mathbf{k} \mapsto \mathbf{k}^\alpha M(\mathbf{k})$ *is an* $L^p$*-multiplier for all* $|\alpha| = m$.

Let $X$ be a UMD space and $A$ be a closed linear operator in $X$. With $n \in \mathbb{N}$ and $\nu \in \mathbb{C}^n$ we consider the boundary value problem in $\mathcal{Q}_n$ given by

$$\begin{array}{rcll} \mathcal{A}(D)u & = & f & (x \in \mathcal{Q}_n), \\ (D^\beta u)|_{x_j=2\pi} - e^{2\pi\nu_j}(D^\beta u)|_{x_j=0} & = & 0 & (j = 1, \dots, n;\ |\beta| < m_1). \end{array} \tag{3.1}$$

In view of the boundary conditions, we refer to the boundary value problem (3.1) as $\nu$-periodic. Here

$$\mathcal{A}(D) := P(D) + Q(D)A := \sum_{|\alpha|\le m_1} p_\alpha D^\alpha + \sum_{|\alpha|\le m_2} q_\alpha D^\alpha A$$

with $m_1, m_2 \in \mathbb{N}$, $m_2 \le m_1$, and $p_\alpha, q_\alpha \in \mathbb{C}$. In what follows, with $m := m_1$ we frequently write $\mathcal{A}(D) = \sum_{|\alpha|\le m}(p_\alpha D^\alpha + q_\alpha D^\alpha A)$ where additional coefficients $q_\alpha$, that is, where $m_2 < |\alpha| \le m_1$, are understood to be equal to zero. Besides that we define the complex polynomials $P(z) := \sum_{|\alpha|\le m_1} p_\alpha z^\alpha$ and $Q(z) := \sum_{|\alpha|\le m_2} q_\alpha z^\alpha$ for $z \in \mathbb{C}^n$.

*Definition* 3.4. A solution of the boundary value problem (3.1) is understood as a function $u \in W^{m_1,p}_{\nu,per}(\mathcal{Q}_n, X) \cap W^{m_2,p}(\mathcal{Q}_n, D(A))$ such that $\mathcal{A}(D)u(x) = f(x)$ for almost every $x \in \mathcal{Q}_n$.

*Remark* 3.5. Since the trace operator with respect to one direction and tangential derivation commute, the $\nu$-periodic boundary conditions as imposed in (3.1) are equivalent to

$$(D_j^\ell u)|_{x_j=2\pi} - e^{2\pi\nu_j}(D_j^\ell u)|_{x_j=0} = 0 \quad (j = 1, \ldots, n,\ 0 \le \ell < m_1).$$

Recall that existence of $(\lambda + \mu A)^{-1}$ implies both $\mu \neq 0$ and $\lambda \in \rho(-\mu A)$.

**Theorem 3.6.** *Let $1 < p < \infty$, and assume $P$ and $Q$ to be elliptic. Then the following assertions are equivalent:*

(i) *For each $f \in L^p(\mathcal{Q}_n, X)$ there exists a unique solution of* (3.1).

(ii) $\big(P(\mathbf{k} - i\nu) + Q(\mathbf{k} - i\nu)A\big)^{-1}$ *exists for $\mathbf{k} \in \mathbb{Z}^n$ and*

$$M_\alpha(\mathbf{k}) := \mathbf{k}^\alpha \big(P(\mathbf{k} - i\nu) + Q(\mathbf{k} - i\nu)A\big)^{-1}$$

*defines a Fourier multiplier for every $|\alpha| = m_1$.*

(iii) $\big(P(\mathbf{k} - i\nu) + Q(\mathbf{k} - i\nu)A\big)^{-1}$ *exists for $\mathbf{k} \in \mathbb{Z}^n$ and for all $|\alpha| = m_1$ there exists a finite subset $G \subset \mathbb{Z}^n$ such that the sets $\{M_\alpha(\mathbf{k});\ \mathbf{k} \in \mathbb{Z}^n \setminus G\}$ are $\mathcal{R}$-bounded.*

*Proof.* (i) $\Rightarrow$ (ii): Let $f \in L^p(\mathcal{Q}_n, X)$ be arbitrary and let $u$ be a solution of (3.1) with right-hand side $e^{\nu\cdot} f$. Then $e^{-\nu\cdot}\mathcal{A}(D)u = f$.

To compute the Fourier coefficients, we first remark that

$$\big(e^{-\nu\cdot} P(D)u\big)\hat{}(\mathbf{k}) = P(\mathbf{k} - i\nu)(e^{-\nu\cdot} u)\hat{}(\mathbf{k})$$

by Lemma 3.2. Concerning $e^{-\nu\cdot}Q(D)Au$, note that by definition of a solution we have $Au \in W^{m_2,p}(\mathcal{Q}_n, X)$. Due to the closedness of $A$, we obtain $D^\alpha A u = AD^\alpha u$ for $|\alpha| \le m_2$, and consequently $Au \in W^{m_2,p}_{\nu,per}(\mathcal{Q}_n, X)$. Now we can apply Lemma 3.2 to see

$$\big(e^{-\nu\cdot} Q(D)Au\big)\hat{}(\mathbf{k}) = Q(\mathbf{k} - i\nu)(e^{-\nu\cdot} Au)\hat{}(\mathbf{k}) = Q(\mathbf{k} - i\nu)A(e^{-\nu\cdot} u)\hat{}(\mathbf{k}).$$

Writing $\mathbf{k}_\nu := \mathbf{k} - i\nu$ for short, we obtain

$$\big(P(\mathbf{k}_\nu) + Q(\mathbf{k}_\nu)A\big)\big(e^{-\nu\cdot} u\big)\hat{}(\mathbf{k}) = \hat{f}(\mathbf{k}). \tag{3.2}$$

For arbitrary $y \in X$ and $\mathbf{k} \in \mathbb{Z}^n$, the choice $f := e^{i\mathbf{k}\cdot} y$ shows $\big(P(\mathbf{k}_\nu) + Q(\mathbf{k}_\nu)A\big)$ to be surjective. Let $z \in D(A)$ such that $\big(P(\mathbf{k}_\nu) + Q(\mathbf{k}_\nu)A\big)z = 0$. For fixed $\mathbf{k} \in \mathbb{Z}^n$ set $v := e^{i\mathbf{k}\cdot} z$ and $u := e^{\nu\cdot} v$. Then

$$P(\mathbf{k}_\nu)\big(e^{-\nu\cdot} u\big)\hat{}(\mathbf{k}) + Q(\mathbf{k}_\nu)A\big(e^{-\nu\cdot} u\big)\hat{}(\mathbf{k}) = 0.$$

As $(e^{-\nu\cdot} u)\hat{}(\mathbf{m}) = 0$ for all $\mathbf{m} \neq \mathbf{k}$, this gives $\mathcal{A}(D)u = 0$, hence $v = u = 0$ and $z = 0$.

Altogether we have shown bijectivity of $P(\mathbf{k}_\nu) + Q(\mathbf{k}_\nu)A$ for $\mathbf{k} \in \mathbb{Z}^n$. The closedness of $A$ yields $\big(P(\mathbf{k}_\nu) + Q(\mathbf{k}_\nu)A\big)^{-1} \in \mathcal{L}(X)$.

For $f \in L^p(\mathcal{Q}_n, X)$ let $u$ be a solution of (3.1) with right-hand side $e^{\nu\cdot}f$ and $v := e^{-\nu\cdot}u$. Then $v \in W^{m_1,p}_{per}(\mathcal{Q}_n, X)$, and (3.2) implies

$$\hat{v}(\mathbf{k}) = \big(P(\mathbf{k}_\nu) + Q(\mathbf{k}_\nu)A\big)^{-1}\hat{f}(\mathbf{k}).$$

This shows

$$M_0 : \mathbb{Z}^n \to \mathcal{L}(L^p(\mathcal{Q}_n, X)); \ \mathbf{k} \mapsto \big(P(\mathbf{k}_\nu) + Q(\mathbf{k}_\nu)A\big)^{-1}$$

to be a Fourier multiplier such that $T_{M_0}$ maps $L^p(\mathcal{Q}_n, X)$ into $W^{m_1,p}_{per}(\mathcal{Q}_n, X)$. Due to Lemma 3.3, we have that $M_\alpha$ is a Fourier multiplier for all $|\alpha| = m_1$.

(ii) $\Rightarrow$ (iii): This follows as in [AB02, Prop. 1.11].

(iii) $\Rightarrow$ (i): For $\mathbf{k} \neq \mathbf{0}$ it holds that

$$\begin{aligned} &P(\mathbf{k}_\nu)\big(P(\mathbf{k}_\nu) + Q(\mathbf{k}_\nu)A\big)^{-1} \\ &\qquad = \frac{P(\mathbf{k}_\nu)}{\sum_{j=1}^n \mathbf{k}^{m_1\mathbf{e}_j}}\left(\sum_{j=1}^n \mathbf{k}^{m_1\mathbf{e}_j}\big(P(\mathbf{k}_\nu) + Q(\mathbf{k}_\nu)A\big)^{-1}\right)\end{aligned}$$

and as there exists $C > 0$ such that $|P(\mathbf{k}_\nu)| \leq C|\sum_{j=1}^n \mathbf{k}^{m_1\mathbf{e}_j}|$ for $\mathbf{k} \in \mathbb{Z}^n \setminus G$ with suitably chosen finite $G \subset \mathbb{Z}^n$, Lemma 2.3 shows that the set

$$\Big\{P(\mathbf{k}_\nu)\big(P(\mathbf{k}_\nu) + Q(\mathbf{k}_\nu)A\big)^{-1} : \ \mathbf{k} \in \mathbb{Z}^n \setminus G\Big\}$$

is $\mathcal{R}$-bounded as well. By Proposition 2.9 it follows that $M_\alpha$ for $|\alpha| = m_1$ as well as $P(\cdot - i\nu)M_0$ are Fourier multipliers. For arbitrary $f \in L^p(\mathcal{Q}_n, X)$ we therefore get $v := T_{M_0}(e^{-\nu\cdot}f) \in W^{m_1,p}_{per}(\mathcal{Q}_n, X)$. As

$$\begin{aligned} &Q(\mathbf{k}_\nu)A\big(P(\mathbf{k}_\nu) + Q(\mathbf{k}_\nu)A\big)^{-1} \\ &\qquad = \mathrm{id}_X - P(\mathbf{k}_\nu)\big(P(\mathbf{k}_\nu) + Q(\mathbf{k}_\nu)A\big)^{-1},\end{aligned} \tag{3.3}$$

$Q(\cdot - i\nu)AM_0$ is a Fourier multiplier, too. By ellipticity of $Q$ and Lemma 2.3 again, the same holds for $\mathbf{k}^\alpha A\big(P(\mathbf{k}_\nu) + Q(\mathbf{k}_\nu)A\big)^{-1}$, $|\alpha| \leq m_2$.

Set $u := e^{\nu\cdot}v = e^{\nu\cdot}T_{M_0}e^{-\nu\cdot}f$. Then $u$ solves (3.1) by construction, and Lemma 3.3 yields $u \in W^{m_1,p}_{\nu,per}(\mathcal{Q}_n, X)$ and $Au \in W^{m_2,p}_{\nu,per}(\mathcal{Q}_n, X)$. Finally, uniqueness of $u$ follows immediately from the uniqueness of the representation as a Fourier series. $\square$

*Remark* 3.7. We have seen in the proof that if one of the equivalent conditions in Theorem 3.6 is satisfied, we have $Au \in W^{m_2,p}_{\nu,per}(\mathcal{Q}_n, X)$. In particular, we get

$$(D^\beta Au)|_{x_j=2\pi} - e^{2\pi\nu_j}(D^\beta Au)|_{x_j=0} = 0 \quad (j = 1, \dots, n; \ |\beta| < m_2)$$

as additional boundary conditions in (3.1).

Theorem 3.6 enables us to treat Dirichlet-Neumann type boundary conditions on $\tilde{\mathcal{Q}}_n := (0,\pi)^n$ for symmetric operators, provided $P$ and $Q$ are of appropriate structure. More precisely, we call a differential operator $\mathcal{A}(D) = \sum_{|\alpha|\leq m}(p_\alpha D^\alpha + q_\alpha D^\alpha A)$ symmetric if for all $|\alpha| \leq m$ either $p_\alpha = q_\alpha = 0$ or $\alpha \in 2\mathbb{N}_0^n$. In particular, $m_1$ is even. As examples, the operators $\mathcal{A}(D_t) := D_t^2 + A$ and $\mathcal{A}(D_1, D_2) := (D_1^2 + D_2^2)^2 + (D_1^4 + D_2^4)A$ are symmetric and satisfy the conditions on $P$ and $Q$ from Theorem 3.6.

In each direction $j \in \{1, \dots, n\}$, we will consider one of the following boundary conditions:

(i) $D_j^\ell u|_{x_j=0} = D_j^\ell u|_{x_j=\pi} = 0 \quad (\ell = 0, 2, \dots, m_1 - 2)$,
(ii) $D_j^\ell u|_{x_j=0} = D_j^\ell u|_{x_j=\pi} = 0 \quad (\ell = 1, 3, \dots, m_1 - 1)$,
(iii) $D_j^\ell u|_{x_j=0} = D_j^{\ell+1} u|_{x_j=\pi} = 0 \quad (\ell = 0, 2, \dots, m_1 - 2)$,
(iv) $D_j^{\ell+1} u|_{x_j=0} = D_j^\ell u|_{x_j=\pi} = 0 \quad (\ell = 0, 2, \dots, m_1 - 2)$.

Note that for a second-order operator, (i) is of Dirichlet type, (ii) is of Neumann type, and (iii) and (iv) are of mixed type. For instance, in case (iii) we have $u|_{x_j=0} = 0$ and $D_j u|_{x_j=\pi} = 0$. Therefore, we refer to these boundary conditions as conditions of Dirichlet-Neumann type. Note that the types may be different in different directions.

**Theorem 3.8.** *Let $\mathcal{A}(D)$ be symmetric, with $P$ and $Q$ being elliptic, and let the boundary conditions be of Dirichlet-Neumann type as explained above. Define $\nu \in \mathbb{C}^n$ by setting $\nu_j := 0$ in cases (i) and (ii) and $\nu_j := i/2$ in cases (iii) and (iv). If for this $\nu$ one of the equivalent conditions of Theorem 3.6 is fulfilled, then for each $f \in L^p(\tilde{\mathcal{Q}}_n, X)$ there exists a unique solution $u \in W^{m,p}(\tilde{\mathcal{Q}}_n, X)$ of $\mathcal{A}(D)u = f$ satisfying the boundary conditions.*

*Proof.* Following an idea from [AB02], the solution is constructed by a suitable even or odd extension of the right-hand side from $(0, \pi)^n$ to $(-\pi, \pi)^n$. For simplicity of notation, let us first consider the case $n = 2$ and boundary conditions of type (ii) in direction $x_1$ and of type (iii) in direction $x_2$. By definition, this leads to $\nu_1 = 0$ and $\nu_2 = \frac{i}{2}$.

Let $f \in L^p(\tilde{\mathcal{Q}}_2, X)$ be arbitrary. First considering the even extension of $f$ to the rectangle $(-\pi, \pi) \times (0, \pi)$ and afterwards its odd extension to $(-\pi, \pi) \times (-\pi, \pi)$, we end up with a function $F$ which fulfills $F(x_1, x_2) = F(-x_1, x_2)$ as well as $F(x_1, x_2) = -F(x_1, -x_2)$ a.e. in $(-\pi, \pi)^2$.

Now we can apply Theorem 3.6 with $\nu = (\nu_1, \nu_2)^T$ as above. (Here and in the following, the result of Theorem 3.6 has to be shifted from the interval $(0, 2\pi)^n$ to the interval $(-\pi, \pi)^n$.) This yields a unique solution $U$ of

$$\begin{array}{rcll} \mathcal{A}(D)U & = & F & \text{in } (-\pi,\pi)\times(-\pi,\pi), \\ D_1^\ell U|_{x_1=-\pi} & = & D_1^\ell U|_{x_1=\pi} & (\ell = 0, \dots, m_1 - 1), \\ -D_2^\ell U|_{x_2=-\pi} & = & D_2^\ell U|_{x_2=\pi} & (\ell = 0, \dots, m_1 - 1). \end{array} \tag{3.4}$$

Symmetry of $\mathcal{A}(D)$ now shows that $V_1(x_1, x_2) := U(-x_1, x_2)$ and $V_2(x_1, x_2) := -U(x_1, -x_2)$ $(x \in (-\pi, \pi)^2)$ are solutions of (3.4) as well. By uniqueness, $V_1 = U = V_2$ follows.

Hence $U_{x_2} := U(\cdot, x_2) \in W^{m,p}((-\pi, \pi), X) \subset C^{m-1}((-\pi, \pi), X)$ for a.e. $x_2 \in (-\pi, \pi)$ is even. Together with symmetry of $U_{x_2}$ due to (3.4), this yields

$$U_{x_2}^{(\ell)}(0) = U_{x_2}^{(\ell)}(\pi) = 0 \quad (\ell = 1, 3, \dots, m_1 - 1.).$$

Accordingly for a.e. $x_1 \in (-\pi, \pi)$ we have that $U_{x_1}$ is odd, and antisymmetry due to (3.4) gives

$$U_{x_1}^{(\ell)}(0) = U_{x_1}^{(\ell+1)}(\pi) = 0 \quad (\ell = 0, 2, \dots, m_1 - 2).$$

Therefore, $u := U|_{(0,\pi)^2}$ solves $\mathcal{A}(D)u = f$ with boundary conditions (ii) for $j = 1$ and (iii) for $j = 2$.

For arbitrary $n \in \mathbb{N}$ and arbitrary boundary conditions of Dirichlet-Neumann type, the construction of the solution follows the same lines. Here we choose even extensions in the cases (ii) and (iv) and odd extensions in the cases (i) and (iii).

On the other hand, let $u$ be a solution of $\mathcal{A}(D)u = f$ satisfying boundary conditions of Dirichlet-Neumann type. We extend $u$ and $f$ to $U$ and $F$ on $(-\pi,\pi)^n$ as described above. Then $U \in W^{m,p}((-\pi,\pi)^n, X)$, $Q(D)AU \in L^p((-\pi,\pi)^n, X)$ and due to symmetry of $\mathcal{A}(D)$ we see that, apart from a shift, $U$ solves (3.1) with right-hand side $F$ and $\nu$ defined as above. Thus, uniqueness of $U$ yields uniqueness of $u$ and the proof is complete. $\square$

*Remark* 3.9. In case $n = 1$ ellipticity of $P$ does no longer force $P$ to be of even order. Hence the same results can be achieved if $\mathcal{A}(D)$ is antisymmetric in the obvious sense, e.g. $\mathcal{A}(D_t) := D_t^3 + D_t + D_t A$.

## 4. Maximal regularity of cylindrical boundary value problems with $\nu$-periodic boundary conditions

Let $F$ be a UMD space and let $\Omega := \mathcal{Q}_n \times V \subset \mathbb{R}^{n+n_V}$ with $V \subset \mathbb{R}^{n_V}$. For $x \in \Omega$ we write $x = (x^1, x^2) \in \mathcal{Q}_n \times V$, whenever we want to refer to the cylindrical geometry of $\Omega$. Accordingly, we write $\alpha = (\alpha^1, \alpha^2) \in \mathbb{N}_0^n \times \mathbb{N}_0^{n_V}$ for a multiindex $\alpha \in \mathbb{N}_0^{n+n_V}$ and $D^\alpha = D^{(\alpha^1,\alpha^2)} =: D_1^{\alpha^1} D_2^{\alpha^2}$.

In the sequel we investigate the vector-valued parabolic initial boundary value problem

$$
\begin{aligned}
u_t + \mathcal{A}_\delta(x,D)u &= f \quad (t \in J,\, x \in \mathcal{Q}_n \times V),\\
B_j(x,D)u &= 0 \quad (t \in J,\, x \in \mathcal{Q}_n \times \partial V,\, j = 1,\dots,m_V),\\
(D^\beta u)|_{x_j=2\pi} - e^{2\pi\nu_j}(D^\beta u)|_{x_j=0} &= 0 \quad (j = 1,\dots,n;\ |\beta| < m_1),\\
u(0,x) &= u_0(x) \quad (x \in \mathcal{Q}_n \times V).
\end{aligned}
\tag{4.1}
$$

Here $J := [0,T)$, $0 < T \le \infty$, denotes a time interval, and the differential operator $\mathcal{A}_\delta(x,D)$ has the form

$$
\begin{aligned}
\mathcal{A}_\delta(x,D) &= P(x^1,D_1) + Q_\delta(D_1)A_V(x^2,D_2)\\
&:= P(x^1,D_1) + \big(Q(D_1)+\delta\big)A_V(x^2,D_2)\\
&:= \sum_{|\alpha^1|\le m_1} p_{\alpha^1}(x^1)D_1^{\alpha^1} + \sum_{|\alpha^1|\le m_2} q_{\alpha^1} D_1^{\alpha^1} A_V(x^2,D_2) + \delta A_V(x^2,D_2)
\end{aligned}
$$

where $\delta \ge 0$ is to be specified. The operator $A_V(x^2,D_2)$ is assumed to be of order $2m_V$ and is augmented with boundary conditions

$$
B_j(x,D) = B_j(x^2,D_2) \quad (j = 1,\dots,m_V)
$$

with operators $B_j(x^2,D_2)$ of order $m_j < 2m_V$ acting on the boundary of $V$. We want to restrict ourselves to $\nu = 0$ or purely imaginary components of $\nu$, since then $i\nu \in \mathbb{R}^n$. In view of the boundary conditions it is sufficient to consider $\nu \in i(-1,1)^n$. Note that periodic as well as antiperiodic boundary conditions are still captured.

This class of equations fits into the framework of Section 3 if we define the operator $A = A_V$ in Section 3 as the $L^p$-realization of the boundary value problem $((A_V(x^2, D_2), B_1(x^2, D_2), \dots, B_{m_V}(x^2, D_2))$. More precisely, for $1 < p < \infty$ we define the operator $A_V$ in $L^p(V, F)$ by

$$\begin{aligned} D(A_V) &:= \{u \in W^{2m,p}(V,F) : B_j(x^2, D_2)u = 0 \ (j = 1, \dots, m_V)\}, \\ A_V u &:= A_V(x, D)u := A_V(x^2, D_2)u \quad (u \in D(A_V)). \end{aligned}$$

Throughout this section, we will assume that the boundary value problem $(A_V, B_1, \dots, B_{m_V})$ satisfies standard smoothness and parabolicity assumptions as, e.g., given in [DHP03, Theorem 8.2]. In particular, $V$ is assumed to be a domain with compact $C^{2m_V}$-boundary, and $(A_V, B_1, \dots, B_{m_V})$ is assumed to be parameter-elliptic with angle $\varphi_{A_V} \in [0, \pi)$. For the notion of parameter-ellipticity of a boundary value problem, we refer to [DHP03, Section 8.1].

Recall that a densely defined operator $A$ is called $\mathcal{R}$-sectorial if there exists a $\theta \in (0, \pi)$ such that

$$\mathcal{R}\big(\{\lambda(\lambda + A)^{-1} : \lambda \in \Sigma_{\pi-\theta}\}\big) < \infty. \tag{4.2}$$

For an $\mathcal{R}$-sectorial operator, $\phi_A^{\mathcal{R}} := \inf\{\theta \in (0, \pi) : \text{(4.2) holds}\}$ is called the $\mathcal{R}$-angle of $A$ (see [DHP03, p. 42]). We mention that we do not impose injectivity for $\mathcal{R}$-sectoriality of an operator $A$. In view of time-dependend problems, $\mathcal{R}$-sectoriality of an operator is closely related to maximal regularity. Recall that a closed and densely defined operator in a Banach space $X$ has maximal $L^q$-regularity if for each $f \in L^q((0, \infty), X)$ there exists a unique solution $w\colon (0, \infty) \to D(A)$ of the Cauchy problem

$$\begin{aligned} w_t + Aw &= f \quad \text{in } (0, \infty), \\ w(0) &= 0 \end{aligned}$$

satisfying the estimate

$$\|w_t\|_{L^q((0,\infty),X)} + \|Aw\|_{L^q((0,\infty),X)} \le C\|f\|_{L^q((0,\infty),X)}$$

with a constant $C$ independent of $f$. By a well-known result due to Weis [Wei01, Thm. 4.2], $\mathcal{R}$-sectoriality in a UMD space with $\mathcal{R}$-angle less than $\frac{\pi}{2}$ is equivalent to maximal $L^q$-regularity for all $1 < q < \infty$. In [DHP03] it was shown that standard parameter-elliptic problems lead to $\mathcal{R}$-sectorial operators:

**Proposition 4.1** ([DHP03, Theorem 8.2]). *Under the assumptions above, for each $\phi > \varphi_{A_V}$ there exists a $\delta_V = \delta_V(\phi) \ge 0$ such that $A_V + \delta_V$ is $\mathcal{R}$-sectorial with $\mathcal{R}$-angle $\phi^{\mathcal{R}}_{A_V+\delta_V} \le \phi$. Moreover,*

$$\mathcal{R}(\{\lambda^{1-\frac{|\alpha^2|}{2m_V}} D^{\alpha^2}(\lambda + A_V + \delta_V)^{-1};\ \lambda \in \Sigma_{\pi-\phi},\ 0 \le |\alpha^2| \le 2m_V\}) < \infty. \tag{4.3}$$

We will show that under suitable assumptions on $P$ and $Q$, $\mathcal{R}$-sectoriality of $A_V$ implies $\mathcal{R}$-sectoriality of the operator related to the cylindrical problem (4.1). For this consider the resolvent problem corresponding to (4.1) which is given by

$$\begin{aligned} \lambda u + \mathcal{A}_\delta(x, D)u &= f \quad (x \in \mathcal{Q}_n \times V), \\ B_j(x, D)u &= 0 \quad (x \in \mathcal{Q}_n \times \partial V,\ j = 1, \dots, m_V), \\ (D^\beta u)|_{x_j=2\pi} - e^{2\pi\nu_j}(D^\beta u)|_{x_j=0} &= 0 \quad (j = 1, \dots, n,\ |\beta| < m_1). \end{aligned} \tag{4.4}$$

For sake of readability, we assume that $m_1 = 2m_V$. The $L^p(\Omega, F)$-realization of the boundary value problem (4.4) is defined as

$$\begin{aligned} D(\mathbb{A}_\delta) &:= \{u \in W^{m_1,p}(\Omega, F) \cap W^{m_1,p}_{\nu,per}(\mathcal{Q}_n, L^p(V, F)) : \\ &\qquad B_j(x, D)u = 0 \ (j = 1, ..., m_V), \quad A_V(x, D)u \in W^{m_2,p}(\mathcal{Q}_n, L^p(V, F))\}, \\ \mathbb{A}_\delta u &:= \mathcal{A}_\delta(x, D)u \quad (u \in D(\mathbb{A}_\delta)). \end{aligned}$$

*Remark* 4.2. a) Since $m_2 \leq m_1$ it holds that

$$D(\mathbb{A}_\delta) = W^{m_1,p}(\Omega, F) \cap W^{m_1,p}_{\nu,per}(\mathcal{Q}_n, L^p(V, F)) \cap W^{m_2,p}(\mathcal{Q}_n, D(A_V)).$$

b) The following techniques apply as well to equations with mixed orders $m_1 \neq 2m_V$. Then, in the definition of $D(\mathbb{A}_\delta)$, the space $W^{m_1,p}(\Omega, F)$ has to be replaced by $\{u \in L^p(\Omega, F) : \ D^\alpha u \in L^p(\Omega, F) \text{ for } \frac{|\alpha^1|}{m_1} + \frac{|\alpha^2|}{2m_V} \leq 1\}$.

4.1. **Constant coefficients.** As it is assumed for $Q(D_1)$ throughout this section we first assume $P(x^1, D_1) = P(D_1)$ to have constant coefficients as well and set

$$\mathcal{A}_{\delta,0} := \mathcal{A}_{\delta,0}(x^2, D) := P(D_1) + Q_\delta(D_1)\big(A_V + \delta_V\big).$$

With $\mathbb{A}_{\delta,0}u := \mathcal{A}_{\delta,0}u$ for $u \in D(\mathbb{A}_{\delta,0}) := D(\mathbb{A}_\delta)$ we formally get $(\lambda + \mathbb{A}_{\delta,0})^{-1} = e^{\nu\cdot}T_{M_\lambda}e^{-\nu\cdot}$ where $T_{M_\lambda}$ denotes the associated operator to

$$M_\lambda(\mathbf{k}) := \big(\lambda + P(\mathbf{k} - i\nu) + Q_\delta(\mathbf{k} - i\nu)(A_V + \delta_V)\big)^{-1}.$$

More generally, the Leibniz rule shows

$$D^\alpha(\lambda + \mathbb{A}_{\delta,0})^{-1} = D^\alpha e^{\nu\cdot}T_{M_\lambda}e^{-\nu\cdot} = \sum_{\beta\leq\alpha} g_\beta(\nu)e^{\nu\cdot}T_{M^\beta_\lambda}e^{-\nu\cdot},$$

where $g_\beta$ is a polynomial depending on $\beta$. Here $T_{M^\beta_\lambda}$ denotes the associated operator to

$$M^\beta_\lambda(\mathbf{k}) := \mathbf{k}^{\beta_1}D^{\beta_2}\big(\lambda + P(\mathbf{k} - i\nu) + Q_\delta(\mathbf{k} - i\nu)(A_V + \delta_V)\big)^{-1}$$

where $\beta = (\beta_1, \beta_2)^T \leq \alpha$. In case $\nu = 0$ we simply have

$$D^\alpha(\lambda + \mathbb{A}_{\delta,0})^{-1} = T_{M^\alpha_\lambda}.$$

**Theorem 4.3.** *Let $1 < p < \infty$, let $F$ be a UMD space enjoying property $(\alpha)$, let $\nu \in i(-1, 1)^n$ and let the boundary value problem $(A_V, B)$ fulfill the conditions of* [DHP03, Theorem 8.2] *with angle of parameter-ellipticity $\varphi_{A_V}$.*

*For $P$ and $Q$ assume that*

(i) *$P$ is homogeneous and parameter-elliptic with angle $\varphi_P \in [0, \pi)$,*
(ii) *$Q$ is homogeneous and parameter-elliptic with angle $\varphi_Q \in [0, \pi)$,*
(iii) *$\varphi_P + \varphi_Q + \varphi_{A_V} < \pi$.*

*Set $\varphi_0 := \max\{\varphi_P, \varphi_Q + \varphi_{A_V}\}$.*

*Then for each $\delta > 0$ the $L^p$-realization $\mathbb{A}_{\delta,0}$ of $\mathcal{A}_{\delta,0}$ is $\mathcal{R}$-sectorial with $\mathcal{R}$-angle $\phi^{\mathcal{R}}_{\mathbb{A}_{\delta,0}} \leq \varphi_0$. Moreover, for each $\phi > \varphi_0$ it holds that*

$$\mathcal{R}(\{\lambda^{1-\frac{|\alpha|}{m_1}}D^\alpha(\lambda + \mathbb{A}_{\delta,0})^{-1} : \ \lambda \in \Sigma_{\pi-\phi}, \ \alpha \in \mathbb{N}_0^{n+n_v}, \ 0 \leq |\alpha| \leq m_1\}) < \infty. \tag{4.5}$$

*In particular, if $\varphi_0 < \frac{\pi}{2}$ then $\mathbb{A}_{\delta,0}$ has maximal $L^q$-regularity for every $1 < q < \infty$, i.e., the initial-boundary value problem (4.1) is well-posed in $L^q([0, T), L^p(\Omega, F))$.*

*If $\nu \neq 0$ or $Q \equiv c$, $c \neq 0$ the assertion remains valid for $\delta = 0$.*

*Proof.* Let $\phi > \varphi_0$. Due to conditions (i) - (iii) on $\varphi_P$, $\varphi_Q$ and $\varphi_{A_V}$ there exists $\vartheta > \varphi_{A_V}$ such that

$$\frac{\lambda + P(\xi)}{Q(\xi)} \in \Sigma_{\pi-\vartheta} \quad (\lambda \in \Sigma_{\pi-\phi},\ \xi \in \mathbb{R}^n \setminus \{0\}).$$

First consider $\nu \neq 0$ and $\delta = 0$. Let $\alpha \in \mathbb{N}_0^{n+n_V}$, $0 \leq |\alpha| \leq m_1 = 2m_V$, $\mathbf{0} \leq \beta \leq \alpha$, and $\mathbf{0} \leq \gamma \leq \mathbf{1}$. For sake of convenience we drop the shift of $A_V$, i.e. we assume $\delta_V = 0$. To prove (4.5) we apply Lemma 2.6 in order to calculate $\mathbf{k}^\gamma \Delta^\gamma M^\beta_{\lambda+\delta}(\mathbf{k})$. In what follows we write $\mathbf{k}_\nu := \mathbf{k} - i\nu$ for short again. Recall that $i\nu \in (-1,1)^n \setminus \{0\}$. As in the proof of Proposition 2.9 it suffices to show that

$$\{\lambda^{1-\frac{|\alpha|}{m_1}} \mathbf{k}^\omega \Delta^\omega N(\mathbf{k}) D^{\beta_2} \big(\lambda + P(\mathbf{k}_\nu) + Q(\mathbf{k}_\nu)A_V\big)^{-1} :\ \lambda \in \Sigma_{\pi-\phi},\ \mathbf{k} \in \mathbb{Z}^n\} \tag{4.6}$$

with $N(\mathbf{k}) := \mathbf{k}^{\beta_1}$ and arbitrary $\omega \leq \gamma$,

$$\{\mathbf{k}^\omega \Delta^\omega P(\mathbf{k}_\nu)\big(\lambda + P(\mathbf{k}_\nu) + Q(\mathbf{k}_\nu)A_V\big)^{-1} :\ \lambda \in \Sigma_{\pi-\phi},\ \mathbf{k} \in \mathbb{Z}^n\}, \tag{4.7}$$

with $\mathbf{0} < \omega \leq \gamma$, and

$$\{\mathbf{k}^\omega \Delta^\omega Q(\mathbf{k}_\nu)A_V\big(\lambda + P(\mathbf{k}_\nu) + Q(\mathbf{k}_\nu)A_V\big)^{-1} :\ \lambda \in \Sigma_{\pi-\phi},\ \mathbf{k} \in \mathbb{Z}^n\} \tag{4.8}$$

with $\mathbf{0} < \omega \leq \gamma$ are $\mathcal{R}$-bounded. Due to our assumptions and Proposition 4.1, in particular due to (4.3), for $0 \leq |\beta_2| \leq m_1 = 2m_V$ the set

$$\left\{\left(\frac{\lambda + P(\mathbf{k}_\nu)}{Q(\mathbf{k}_\nu)}\right)^{1-\frac{|\beta_2|}{m_1}} D^{\beta_2}\left(\frac{\lambda + P(\mathbf{k}_\nu)}{Q(\mathbf{k}_\nu)} + A_V\right)^{-1} :\ \lambda \in \Sigma_{\pi-\phi},\ \mathbf{k} \in \mathbb{Z}^n\right\}$$

is $\mathcal{R}$-bounded. For $\beta_2 = 0$ this yields the $\mathcal{R}$-boundedness of

$$\{\big(\lambda + P(\mathbf{k}_\nu)\big)\big(\lambda + P(\mathbf{k}_\nu) + Q(\mathbf{k}_\nu)A_V\big)^{-1} :\ \lambda \in \Sigma_{\pi-\phi},\ \mathbf{k} \in \mathbb{Z}^n\} \tag{4.9}$$

and with it the $\mathcal{R}$-boundedness of

$$\{Q(\mathbf{k}_\nu)A_V\big(\lambda + P(\mathbf{k}_\nu) + Q(\mathbf{k}_\nu)A_V\big)^{-1} :\ \lambda \in \Sigma_{\pi-\phi},\ \mathbf{k} \in \mathbb{Z}^n\}. \tag{4.10}$$

Since $\nu$ is supposed to have at least one non-zero component, there exists $\varepsilon > 0$ such that $|\mathbf{k}_\nu| > \varepsilon$ holds true for all $\mathbf{k} \in \mathbb{Z}^n$. Moreover, there exists $C > 0$ such that

$$\frac{\lambda^{1-\frac{|\alpha|}{m_1}} |\mathbf{k}|^{|\omega|} |\Delta^\omega N(\mathbf{k})| |Q(\mathbf{k}_\nu)|^{1-\frac{|\beta_2|}{m_1}}}{|\lambda + P(\mathbf{k}_\nu)|^{1-\frac{|\beta_2|}{m_1}} |Q(\mathbf{k}_\nu)|} \leq C \quad \text{and} \quad \frac{|\mathbf{k}|^{|\omega|} |\Delta^\omega P(\mathbf{k}_\nu)|}{|\lambda + P(\mathbf{k}_\nu)|} \leq C$$

for all $\mathbf{k} \in \mathbb{Z}^n$ and all $\lambda \in \Sigma_{\pi-\phi}$ due to parameter-ellipticity of $P$ and ellipticity of $Q$. (In case $\delta > 0$, parameter-ellipticity of $Q$ has to be used.)

Again we apply the contraction principle of Kahane to prove (4.6) and (4.7).

Similarly, ellipticity of $Q$ proves (4.8) as well as $D^\alpha A_V(\lambda + \mathbb{A}_{\delta,0})^{-1} f \in L^p(\Omega, F))$ for $|\alpha| \leq m_2$.

Now consider the case $\nu = 0$. Note that the ideas of the first part of the proof carry over to this situation only if $\mathbf{k} \neq 0$. Two $\mathcal{R}$-boundedness statements have to be proven in order to apply the multiplier theorem. First, $\mathcal{R}$-boundedness of

$$\{\lambda^{1-\frac{|\alpha|}{m_1}} M^\alpha_\lambda :\ \lambda \in \Sigma_{\pi-\phi},\ \mathbf{k} \in \mathbb{Z}^n\}.$$

This follows immediately due to homogeneity arguments. Recall the structure of $M_\lambda^\alpha$, in particular the fact that we no longer have to consider $M_\lambda^\beta$ with $|\beta| < |\alpha|$ and that

$$\lambda^{1-\frac{|\alpha|}{m_1}} M_\lambda^\alpha(0) = \begin{cases} 0, & \alpha_1 \neq 0 \\ \lambda^{1-\frac{|\alpha_2|}{m_1}} D^{\alpha_2}\big(\lambda + \delta A_V\big)^{-1}, & \alpha_1 = 0. \end{cases}$$

Second, we have to prove $\mathcal{R}$-boundedness of (4.6), (4.7) and (4.8), this time however with $\mathbf{k} \in \mathbb{Z}^n \setminus \{0\}$ instead of $\mathbf{k} \in \mathbb{Z}^n$. Hence $\mathbf{k} \neq 0$ and part one of the proof applies verbatim.

The last claim on maximal $L^q$-regularity now follows from [Wei01, Thm. 4.2]. □

*Remark* 4.4. We have seen in the proof that $A_V u \in W^{m_2,p}_{\nu,per}(\mathcal{Q}_n, L^p(V,F))$, i.e. the solution $u$ of (4.4) fulfills the further boundary condition

$$(D^\beta A_V u)|_{x_j=2\pi} - e^{2\pi\nu_j}(D^\beta A_V u)|_{x_j=0} = 0 \quad (j = 1,\dots,n;\ |\beta| < m_2)$$

(cf. Remark 3.7). Additionally, we have seen in the proof that

$$\mathcal{R}(\{D^\alpha A_V(\lambda + \mathbb{A}_{\delta,0})^{-1} :\ \lambda \in \Sigma_{\pi-\phi},\ 0 \le |\alpha| \le m_2\}) < \infty. \tag{4.11}$$

Note that the shift $\delta > 0$ cannot be neglected in case $Q \not\equiv c$, $c \in \mathbb{R}$ and $\nu = 0$. To see this, take a right-hand side $f \in L^p(\Omega, F)$ which is given as constant extension of a function in $g \in L^p(V,F) \setminus D(A_V)$. If $\lambda u + \mathcal{A}(D)u = f$, then $\lambda\hat{u}(0) = \hat{f}(0) = g$ by parameter-ellipticity of $P$ and $Q$. Hence $u \notin D(\mathbb{A}_0)$.

*Remark* 4.5. Consider again boundary value problems in $(0,\pi)^n \times V$ with Dirichlet-Neumann type boundary conditions and a symmetric setting with respect to $(0,\pi)^n$. As the extension and restriction operators defined above are bounded, Theorem 3.8 immediately yields the related result for Dirichlet-Neumann type boundary conditions. In particular, we obtain maximal regularity results also for boundary conditions of mixed type (iii) and (iv).

4.2. **Non-constant coefficients of $P$.** In this subsection, $P(x^1, D_1)$ is allowed to have non-constant coefficients, where we assume that

$$\begin{cases} p_{\alpha^1} \in C_{per}(\mathcal{Q}_n) \text{ for } |\alpha^1| = m_1, \\ p_{\alpha^1} \in L^{r_\eta}(\mathcal{Q}_n) \text{ for } |\alpha^1| = \eta < m_1,\ r_\eta \ge p,\ \dfrac{m_1 - \eta}{n-k} > \dfrac{1}{r_\eta}. \end{cases} \tag{4.12}$$

Here $C_{per}(\mathcal{Q}_n) := \{f \in C([0,2\pi]^n) :\ f|_{x_j=0} = f|_{x_j=2\pi}\ (j=1,\dots,n)\}$. However, in order to apply perturbation results similar to [DHP03] or [NS], we assume $Q \equiv 1$, i.e. we consider

$$\mathcal{A}(x,D) := P(x^1, D_1) + A_V(x^2, D_2).$$

**Theorem 4.6.** *Let $1 < p < \infty$, let $F$ be a UMD space enjoying property $(\alpha)$, let $\Omega := \mathcal{Q}_n \times V$, and let the boundary valued problem $(A_V, B)$ fulfill the conditions of* [DHP03, Theorem 8.2] *with angle of parameter-ellipticity $\varphi_{A_V}$.*

*For $P$ assume that*

- *the coefficients satisfy* (4.12) *and*
- *$P$ is parameter-elliptic with angle $\varphi_P \in [0, \pi - \varphi_{A_V})$ uniformly in $x \in \overline{\mathcal{Q}}_n$.*

*Set $\varphi_0 := \max\{\varphi_P, \varphi_{A_V}\}$. Then for each $\phi > \varphi_0$ there exists $\mu = \mu(\phi) \geq 0$ such that the $L^p$-realization $\mathbb{A} + \mu$ of $\mathcal{A} + \mu$ is $\mathcal{R}$-sectorial with $\mathcal{R}$-angle $\phi^{\mathcal{R}}_{\mathbb{A}+\mu} \leq \phi$. Moreover, we have*

$$\mathcal{R}(\{\lambda^{1-\frac{|\alpha|}{m_1}} D^\alpha(\lambda + \mathbb{A} + \mu)^{-1} : \lambda \in \Sigma_{\pi-\phi},\ \alpha \in \mathbb{N}_0^{n+n_v},\ 0 \leq |\alpha| \leq m_1\}) < \infty. \tag{4.13}$$

*In particular, if $\varphi_0 < \frac{\pi}{2}$ then there exists $\mu > 0$ such that $\mathbb{A} + \mu$ has maximal $L^q$-regularity for every $1 < q < \infty$.*

*Proof.* In a first step, we consider $P(x, D)$ to be a homogeneous differential operator with slightly varying coefficients. That is, we consider

$$\mathcal{A}^{va}(x, D) := P_0(D_1) + R(x^1, D_1) + A_V(x^2, D_2),$$

where $P_0(D_1) := \sum_{|\alpha^1|=2m} p_{\alpha^1} D_1^{\alpha^1}$ is assumed to have constant coefficients and $R(x^1, D_1) := \sum_{|\alpha^1|=2m} r_{\alpha^1}(x^1) D_1^{\alpha^1}$ fulfills $\sum_{|\alpha^1|=2m} \|r_{\alpha^1}\|_\infty \leq \eta$ with $\eta > 0$ sufficiently small. Then the claim follows due to perturbation results for $\mathcal{R}$-sectorial operators (see [DHP03], [NS]) from Theorem 4.3.

In a second step, we choose a finite but sufficiently fine open covering of $\mathcal{Q}_n$. In view of the periodicity of the top order coefficients, we may assume every open set of the covering, which intersects with $\mathbb{R}^n \setminus \mathcal{Q}_n$ to be cut at the boundary of $\mathcal{Q}_n$ and continued within $\mathcal{Q}_n$ on the opposite side. By means of reflection and cut-off techniques, this enables us to define local operators with slightly varying coefficients. With the help of a partition of the unity and perturbation results for lower order terms subject to condition (4.12), just as in [NS], the claim follows. $\square$

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