Thermoelasticity with second sound — Exponential stability in linear and nonlinear 1-d

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Dedicated to Professor Rolf Leis on the occasion of his seventieth birthday in 2001

Abstract: We consider linear and nonlinear thermoelastic systems in one space dimension where thermal disturbances are modeled propagating as wave-like pulses traveling at finite speed. This removal of the physical paradox of infinite propagation speed in the classical theory of thermoelasticity within Fourier's law is achieved using Cattaneo's law for heat conduction. For different boundary conditions, in particular for those arising in pulsed laser heating of solids, the exponential stability of the now purely, but slightly damped, hyperbolic linear system is proved. A comparison to classical hyperbolic-parabolic thermoelasticity is given. For Dirichlet type boundary conditions — rigidly clamped, constant temperature — the global existence of small, smooth solutions and the exponential stability are proved for a nonlinear system.

1 Introduction

We consider first the linear equations of thermoelasticity that model the second sound effect. If $u = u(t,x)$, $\theta = \theta(t,x)$ and $q = q(t,x)$ for $t \geq 0$, $x \in (0,L) \subset \mathbb{R}^1$ for some fixed $L > 0$, denote the unknown functions representing the displacement, the temperature difference to a fixed reference temperature, and the heat flux, respectively, the differential equations for u, θ, q are

$$u_{tt} - \alpha u_{xx} + \beta\theta_x = 0, \tag{1.1}$$

$$\theta_t + \gamma q_x + \delta u_{tx} = 0, \tag{1.2}$$

$$\tau_0 q_t + q + \kappa\theta_x = 0, \tag{1.3}$$

where $\alpha, \beta, \gamma, \delta, \tau_0, \kappa$ are positive constants, appropriate for an underlying homogeneous medium. In particular, τ_0 is the relaxation time, a parameter being small in comparison to the others and responsible with the term $\tau_0 q_t$ in (1.3) that we have a hyperbolic system. (1.3) represents Cattaneo's law of heat conduction modeling thermal disturbances as wave-like pulses traveling at finite speed. This way, the well-known physical paradox of infinite propagation speed in the classical thermoelastic theory, or the classical pure theory of heat conduction, where instead

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of Cattaneo's law (1.3) Fourier's law of heat conduction is used, is removed. Fourier's law corresponds to $\tau_0 = 0$ in (1.3), i.e.

$$q + \kappa\theta_x = 0, \tag{1.4}$$

which turns (1.2) into

$$\theta_t - \kappa\theta_{xx} + \delta u_{tx} = 0, \tag{1.5}$$

and (1.1), (1.5) are the "classical" equations of thermoelasticity.
For a discussion of this model second sound see [14, 18, 21, 2], for classical thermoelasticity [1, 10], for instance. In section 6 the (nonlinear) equations are derived.
For the classical system without second sound, i.e. for (1.1), (1.5), it is known for various boundary conditions that solutions to initial boundary value problems are exponentially stable, i.e. they tend to an equilibrium state exponentially fast, in a uniform manner. In other words, the impact of dissipation given through heat conduction, modeled by Fourier's law and connected to the displacement by the differential equations (1.1), (1.5), is strong enough to also damp the displacement in an exponential manner. By the way, this is not true in general in higher dimensions, cp. [11, 12, 9, 10]. Here we address the question whether the even weaker dissipation in a now purely hyperbolic system as (1.1)–(1.3) is still predominating to assure that the whole system, in particular the displacement, is decaying exponentially as time tends to infinity.
Prescribing initial conditions,

$$u(0,\cdot) = u_0, \quad u_t(0,\cdot) = u_1, \quad \theta(0,\cdot) = \theta_0, \quad q(0,\cdot) = q_0, \tag{1.6}$$

we shall give a positive answer to this question for various boundary conditions as there are:

I. A rigidly clamped medium with zero heat flux on the boundary,

$$u(t,0) = u(t,L) = q(t,0) = q(t,L) = 0, \quad t \geq 0. \tag{1.7}$$

II. A rigidly clamped medium with temperature held constant on the boundary,

$$u(t,0) = u(t,L) = \theta(t,0) = \theta(t,L) = 0, \quad t \geq 0. \tag{1.8}$$

III. Mixed boundary conditions: Dirichlet type (1.8) in $x = L$ and free boundary conditions at $x = 0$,

$$\alpha u_x(t,0) - \beta\theta(t,0) = 0, \quad \theta_x(t,0) = 0, \tag{1.9}$$

$$u(t,L) = \theta(t,L) = 0. \tag{1.10}$$

The last boundary conditions arise in pulsed laser heating of solids, for instance in laser assisted particle removal from silicon wafers, cp. [23, 20, 15]. Actually these applications were the reason for our studies observing that there is very little in the literature concerning the asymptotic stability of thermoelastic systems with second sound. One should mention a paper by Sherief [19] where the stability of the null solution also in higher dimension is proved, and the work of Tarabek [22] who studied even nonlinear systems in 1-d and obtained, as a by-product, for the boundary conditions (1.7) the strong convergence of derivatives of solutions to zero. The rate of decay, an exponential stability result, is not given. To prove the latter for various boundary conditions is one of our tasks. It will be necessary to construct suitable Lyapunov functions for each set of boundary conditions and to combine various techniques from energy methods, multiplier approach and boundary control, cp. [16, 17, 10]. It will also be possible to explicitly control the decay rates obtained, cp. the application in pulsed laser heating, and also a comparison to the classical case $\tau_0 = 0$ will be presented. Finally we discuss nonlinear systems under the boundary conditions (1.8), and we shall prove the global existence of small solutions and the exponential stability.

The paper is organized as follows: In section 2 we shall look at the boundary conditions (1.7), prove the exponential stability, control the constants arising in terms of the data $\alpha, \beta, \gamma, \delta, \tau_0$ being the material coefficients, and compare it to the case $\tau_0 = 0$. The exponential stability for the boundary conditions (1.8) and for (1.9), (1.10) is proved in section 3 and 4, respectively. In section 5 we provide arguments for showing that the two systems, either $\tau_0 > 0$ or $\tau_0 = 0$, are close to each other, in the sense of energy estimates, of order τ_0^2. In section 6 we derive the (nonlinear) equations and present the local well-posedness under the boundary condition (1.8). A global existence theorem is proved in section 7 together with the exponential stability. Finally and for completeness we discuss the well-posedness of the initial boundary value problem in section 8 in a semigroup setting, with less assumption than in the nonlinear well-posedness result from section 6.

2 Rigidly clamped and zero heat flux

First we consider the differential equations

$$u_{tt} - \alpha u_{xx} + \beta \theta_x = 0, \tag{2.1}$$

$$\theta_t + \gamma q_x + \delta u_{tx} = 0, \tag{2.2}$$

$$\tau_0 q_t + q + \kappa \theta_x = 0 \tag{2.3}$$

with initial conditions

$$u(0, \cdot) = u_0, \quad u_t(0, \cdot) = u_1, \quad \theta(0, \cdot) = \theta_0, \quad q(0, \cdot) = q_0, \tag{2.4}$$

and the boundary conditions modeling a rigidly clamped medium with zero heat flux on the boundary, i.e.

$$u(t, 0) = u(t, L) = q(t, 0) = q(t, L) = 0. \tag{2.5}$$

This boundary condition, also having been studied in [22], is somehow the easiest one with respect to boundary integrals that appear in constructing an appropriate Lyapunov function. While in classical thermoelasticity ($\tau_0 = 0$) the system of differential equations is hyperbolic-parabolic, we observe that (1.1)–(1.3) is purely hyperbolic, but damped hyperbolic. In fact, rewriting (1.1)–(1.3) for $W := (u_x, u_t, \theta, q)$ we see that W satisfies

$$W_t = AW_x + B(W), \tag{2.6}$$

where

$$A := \begin{pmatrix} 0 & 1 & 0 & 0 \\ \alpha & 0 & -\beta & 0 \\ 0 & -\delta & 0 & -\gamma \\ 0 & 0 & -\kappa/\tau_0 & 0 \end{pmatrix}, \qquad B(W) := \begin{pmatrix} 0 \\ 0 \\ 0 \\ -\frac{1}{\tau_0}q \end{pmatrix}.$$

Computing $\det(A-\lambda)$, $\lambda \in \mathbb{C}$, we get

$$\det(A-\lambda) = \lambda^4 - \lambda^2 \left[\frac{\gamma\kappa}{\tau_0} + \beta\delta + \alpha\right] + \frac{\gamma\kappa\alpha}{\tau_0}$$

hence the four real, district eigenvalues of A are

$$\lambda_{1|2|3|4} = \pm\sqrt{\frac{\gamma\kappa}{\tau_0} + \beta\delta + \alpha \pm \sqrt{(\beta\delta + \frac{\gamma\kappa}{\tau_0} - \alpha)^2 + 4\alpha\beta\delta}}.$$

Thus (2.6) is a strictly hyperbolic system in the main part $W_t = AW_x$ with a damping term $B(W)$. The question, which will be answered positively, is if the damping is still strong enough to exponentially stabilize all components.
In the sequel we shall also keep track of the constants showing up in order to get a rough bound for the decay rate and the constants arising in the final estimate in Theorem 2.1 below.

Here and in the following sections 3–5 we shall assume the existence of solutions in the Sobolev spaces we need for our computations. We shall comment on this in section 8, cp. also section 6.

Multiplying (2.1) by $\kappa\delta u_t$, (2.2) by $\kappa\beta\theta$, and (2.3) by $\gamma\beta q$, we obtain after integrating for the first energy term

$$E_1(t) := \frac{1}{2}\int_0^L (\kappa\delta u_t^2 + \kappa\delta\alpha u_x^2 + \kappa\beta\theta^2 + \gamma\beta\tau_0 q^2)(t,x)dx \tag{2.7}$$

the estimate

$$\frac{d}{dt}E_1(t) = -\beta\gamma\int_0^L q^2 dx, \tag{2.8}$$

which demonstrates the dissipative character of the system.

Differentiating (2.1)–(2.3) with respect to t, we get in the same manner

$$\frac{d}{dt}E_2(t) = -\beta\gamma\int_0^L q_t^2 dx, \tag{2.9}$$

where

$$E_2(t) := \frac{1}{2}\int_0^L (\kappa\delta u_{tt}^2 + \kappa\delta\alpha u_{xt}^2 + \kappa\beta\theta_t^2 + \gamma\beta\tau_0 q_t^2)(t,x)dx. \tag{2.10}$$

From (2.3) we conclude

$$\int_0^L \theta_x^2 dx \le 2\frac{\tau_0^2}{\kappa^2}\int_0^L q_t^2 dx + \frac{2}{\kappa^2}\int_0^L q^2 dx. \tag{2.11}$$

Multiplying (2.1) by $\frac{1}{\alpha}u_{xx}$ we get

$$
\begin{aligned}
\int_0^L u_{xx}^2 dx &= -\frac{1}{\alpha}\int_0^L u_{ttx}u_x dx + \frac{\beta}{\alpha}\int_0^L \theta_x u_{xx} dx \\
&\le -\frac{1}{\alpha}\frac{1}{dt}\left(\int_0^L u_{tx}u_x\, dx\right) + \frac{1}{\alpha}\int_0^L u_{tx}^2 dx + \frac{3\beta^2}{4\alpha^2}\int_0^L \theta_x^2 dx + \frac{1}{3}\int_0^L u_{xx}^2 dx,
\end{aligned}
$$

which implies

$$
\frac{2}{3}\int_0^L u_{xx}^2 dx + \frac{1}{\alpha}\frac{d}{dt}\left(\int_0^L u_{tx}u_x dx\right) \le \frac{1}{\alpha}\int_0^L u_{tx}^2 dx + \frac{3\beta^2}{4\alpha^2}\int_0^L \theta_x^2 dx. \tag{2.12}
$$

Multiplying (2.2) by $\frac{3}{\alpha\delta}u_{tx}$ we get

$$
\begin{aligned}
\frac{3}{\alpha}\int_0^L u_{tx}^2 dx &= \frac{3\gamma}{\alpha^2\delta}\frac{d}{dt}\left(\int_0^L q u_{tt} dx\right) + \frac{3\beta\gamma}{\alpha^2\delta}\frac{d}{dt}\left(\int_0^L q\theta_x dx\right) - \frac{3\gamma}{\alpha\delta}\left(\int_0^L q_t u_{xx} dx\right) \\
&\quad + \frac{3}{\alpha\delta}\frac{d}{dt}\left(\int_0^L \theta_x u_t dx\right) - \frac{3}{\delta}\int_0^L \theta_x u_{xx} dx + \frac{3\beta}{\alpha\delta}\int_0^L \theta_x^2 dx \\
&\le \frac{3\gamma}{\alpha^2\delta}\frac{d}{dt}\left(\int_0^L q u_{tt} dx\right) - \frac{3\beta\tau_0\gamma}{\alpha^2\delta\kappa}\frac{d}{dt}\left(\int_0^L q q_t dx\right) - \frac{3\beta\gamma}{\alpha^2\delta\kappa}\frac{d}{dt}\left(\int_0^L q^2 dx\right) \\
&\quad + \frac{1}{12}\int_0^L u_{xx}^2 dx + \frac{27\gamma^2}{\alpha^2\delta^2}\int_0^L q_t^2 dx + \frac{3}{\alpha\delta}\frac{d}{dt}\left(\int_0^L \theta_x u_t dx\right) \\
&\quad + \frac{1}{12}\int_0^L u_{xx}^2 dx + \frac{27}{\delta^2}\int_0^L \theta_x^2 dx + \frac{3\beta}{\alpha\delta}\int_0^L \theta_x^2 dx \\
&= \frac{d}{dt}\left\{\int_0^L \frac{3\gamma}{\alpha^2\delta} q u_{tt} - \frac{3\beta\tau_0\gamma}{\alpha^2\delta\kappa} q q_t - \frac{3\beta\gamma}{\alpha^2\delta\kappa} q^2 - \frac{3\tau_0}{\alpha\delta\kappa} q_t u_t - \frac{3}{\kappa\alpha\delta} q u_t dx\right\} \\
&\quad + \frac{1}{6}\int_0^L u_{xx}^2 dx + \frac{27\gamma^2}{\alpha^2\delta^2}\int_0^L q_t^2 dx + \left(\frac{27}{\delta^2} + \frac{3\beta}{\alpha\delta}\right)\int_0^L \theta_x^2 dx.
\end{aligned} \tag{2.13}
$$

Combining (2.12) and (2.13) we obtain

$$
\begin{aligned}
&\frac{2}{\alpha}\int_0^L u_{tx}^2 dx + \frac{1}{2}\int_0^L u_{xx}^2 dx + \frac{d}{dt}\left\{\int_0^L \frac{1}{\alpha} u_{tx}u_x - \frac{3\gamma}{\alpha^2\delta} q u_{tt} + \frac{3\beta\tau_0\gamma}{\alpha^2\delta\kappa} q q_t \right. \\
&\qquad \left. + \frac{3\beta\gamma}{\alpha^2\delta\kappa} q^2 + \frac{3\tau_0}{\alpha\delta\kappa} q_t u_t + \frac{3}{\kappa\alpha\delta} q u_t dx\right\} \\
&\le \frac{27\gamma^2}{\alpha^2\delta^2}\int_0^L q_t^2 dx + \left(\frac{27}{\delta^2} + \frac{3\beta}{\alpha\delta} + \frac{3\beta^2}{4\alpha^2}\right)\int_0^L \theta_x^2 dx.
\end{aligned} \tag{2.14}
$$

The differential equation (2.2) together with the boundary conditions (2.5) yields

$$\int_0^L \theta(t,x)dx = \int_0^L \theta_0(x)dx, \qquad t \geq 0.$$

Then $\overline{\theta}$ defined by

$$\overline{\theta}(t,x) := \theta(t,x) - \frac{1}{L}\int_0^L \theta_0(x)dx$$

satisfies with u and q the same differential equations (2.1)–(2.3) as (u,θ,q), but additionally we have the first Poincaré inequality

$$\int_0^L v^2(x)dx \leq \frac{L^2}{\pi^2}\int_0^L v_x^2(x)dx \tag{2.15}$$

for $v = \overline{\theta}(t,\cdot)$, as well as for $v = u$ or $v = q$.

In the sequel we shall work with $\overline{\theta}$ but still write θ for simplicity until we shall have proved Theorem 2.1.

Now we conclude from (2.1), (2.3), using (2.15),

$$\int_0^L u_{tt}^2 + u_t^2 + \theta^2 dx \leq 2\alpha^2 \int_0^L u_{xx}^2 dx + \left(2\beta^2 + \frac{L^2}{\pi^2}\right)\frac{2}{\kappa^2}\int_0^L \tau_0^2 q_t^2 + q^2 dx + \frac{L^2}{\pi^2}\int_0^L u_{tx}^2 dx. \tag{2.16}$$

Multiplying (2.1) by u we obtain

$$\alpha \int_0^L u_x^2 dx \leq \frac{L^2}{\pi^2\alpha}\int_0^L u_{tt}^2 dx + \frac{\alpha}{4}\int_0^L u_x^2 dx + \frac{\beta^2 L^2}{\pi^2\alpha}\int_0^L \theta_x^2 dx + \frac{\alpha}{4}\int_0^L u_x^2 dx,$$

whence

$$\frac{\alpha}{2}\int_0^L u_x^2 dx \leq \frac{L^2}{\pi^2\alpha}\int_0^L u_{tt}^2 dx + \frac{\beta^2 L^2}{\pi^2\alpha}\int_0^L \theta_x^2 dx \tag{2.17}$$

follows. Multiplication of (2.2) by θ_t yields

$$\begin{aligned}
\int_0^L \theta_t^2 dx &= -\int_0^L \gamma q_x \theta_t dx - \delta \int_0^L u_{tx}\theta_t dx \\
&\leq \frac{d}{dt}\left(\int_0^L \gamma q \theta_x dx\right) + \frac{\gamma^2}{2}\int_0^L q_t^2 dx + \frac{1}{2}\int_0^L \theta_x^2 dx + \frac{\delta^2}{2}\int_0^L u_{tx}^2 dx + \frac{1}{2}\int_0^L \theta_t^2 dx
\end{aligned}$$

which implies

$$\int_0^L \theta_t^2 dx - \frac{d}{dt}\left(\int_0^L 2\gamma q\theta_x dx\right) \leq \gamma^2 \int_0^L q_t^2 dx + \delta^2 \int_0^L u_{tx}^2 dx + \int_0^L \theta_x^2 dx. \tag{2.18}$$

Now we are able to define the desired Lyapunov function F:

For $\varepsilon > 0,\ \ \mu > 0$, to be determined later on, let

$$
\begin{aligned}
F(t) \quad &:= \quad \frac{1}{\varepsilon}\left(E_1(t)+E_2(t)\right)+\int_0^L \frac{3\beta\gamma}{\alpha^2\delta\kappa}\,q^2dx+\int_0^L\left\{\frac{1}{\alpha}\,u_{tx}u_x-\frac{3\gamma}{\alpha^2\delta}\,qu_{tt}\right.\\
&\quad\left.+\frac{3\beta\tau_0\gamma}{\alpha^2\delta\kappa}\,qq_t+\frac{3\beta\gamma}{\alpha^2\delta\kappa}q^2+\frac{3\tau_0}{\alpha\delta\kappa}\,q_tu_t+\frac{3}{\kappa\alpha\delta}\,qu_t-2\mu\gamma q\theta_x\right\}dx. \qquad (2.19)
\end{aligned}
$$

Then we conclude from (2.8), (2.9), (2.14), (2.18)

$$
\begin{aligned}
\frac{d}{dt}F(t) \quad &\leq \quad -\frac{\beta\gamma}{\varepsilon}\int_0^L(q^2+q_t^2)dx-\frac{2}{\alpha}\int_0^L u_{tx}^2dx-\frac{1}{2}\int_0^L u_{xx}^2dx-\mu\int_0^L\theta_t^2dx\\
&\quad+\left(\frac{27}{\delta^2}+\frac{3\beta}{\alpha\delta}+\frac{3\beta^2}{4\alpha^2}+\mu\right)\int_0^L\theta_x^2dx+\left(\frac{27\gamma^2}{\alpha^2\delta^2}+\mu\gamma^2\right)\int_0^L q_t^2dx\\
&\quad+\mu\delta^2\int_0^L u_{tx}^2dx. \qquad (2.20)
\end{aligned}
$$

From (2.16) we get

$$
\begin{aligned}
&-\mu 2\alpha^2\int_0^L u_{xx}^2dx-\mu\left(\frac{4\beta^2}{\kappa^2}+\frac{2L^2}{\pi^2\kappa^2}\right)\int_0^L\tau_0^2q_t^2+q^2dx-\mu\frac{L^2}{\pi^2}\int_0^L u_{tx}^2dx\\
&\quad\leq\ -\mu\int_0^L u_{tt}^2+u_t^2+\theta^2dx, \qquad (2.21)
\end{aligned}
$$

while (2.17) yields

$$
-\frac{\mu^2L^2}{\pi^2\alpha}\int_0^L u_{tt}^2dx-\frac{2\mu^2\beta^2L^2}{\pi^2\alpha\kappa^2}\int_0^L\tau_0^2q_t^2+q^2dx\leq-\frac{\mu^2\alpha}{2}\int_0^L u_x^2dx. \qquad (2.22)
$$

Combining (2.20)–(2.22) we conclude

$$
\begin{aligned}
\frac{d}{dt}F(t) \quad &\leq \quad -\int_0^L q^2\left[\frac{\gamma\beta}{\varepsilon}-\frac{2}{\kappa^2}\left(\frac{27}{\delta^2}+\frac{3\beta}{\alpha\delta}+\frac{3\beta^2}{4\alpha^2}+\mu\right)\right.\\
&\quad\left.-\mu\left(\frac{4\beta^2}{\kappa^2}+\frac{2L^2}{\pi^2\kappa^2}\right)-\frac{2\mu^2\beta^2L^2}{\pi^2\alpha\kappa^2}\right]dx\\
&\quad-\int_0^L q_t^2\left[\frac{\gamma\beta}{\varepsilon}-\frac{2\tau_0^2}{\kappa^2}\left(\frac{27}{\delta^2}+\frac{3\beta}{\alpha\delta}+\frac{3\beta^2}{4\alpha^2}+\mu\right)-\frac{27\gamma^2}{\alpha^2\delta^2}-\mu\gamma^2\right.\\
&\quad\left.-\mu\tau_0^2\left(\frac{4\beta^2}{\kappa^2}+\frac{2L^2}{\pi^2\kappa^2}\right)-\frac{2\tau_0^2\mu^2\beta^2L^2}{\pi^2\alpha\kappa^2}\right]dx\\
&\quad-\int_0^L u_{tx}^2\left[\frac{2}{\alpha}-\frac{\mu L^2}{\pi^2}-\mu\delta^2\right]dx-\int_0^L u_{xx}^2\left[\frac{1}{2}-2\mu\alpha^2\right]dx
\end{aligned}
$$

$$-\mu\int_0^L \theta_t^2 dx - \int_0^L u_{tt}^2\left[\mu - \frac{\mu^2 L^2}{\pi^2\alpha}\right]dx - \mu\int_0^L u_t^2 dx - \mu\int_0^L \theta^2 dx$$
$$-\frac{\alpha}{2}\mu^2\int_0^L u_x^2 dx. \tag{2.23}$$

We choose $1 \geq \mu = \varepsilon > 0$ such that all terms on the right-hand side of (2.23) become negative:

$$\varepsilon \leq \frac{\pi^2\alpha}{2L^2},\ \varepsilon \leq \frac{1}{4\alpha^2},\ \varepsilon \leq \frac{\pi^2}{(L^2+\pi^2\delta^2)\alpha},$$

$$\varepsilon \leq \frac{\gamma\beta}{2}\left\{\frac{2\tau_0^2}{\kappa^2}\left(\frac{27}{\delta^2}+\frac{3\beta}{\alpha\delta}+\frac{3\beta^2}{4\alpha^2}+1\right)+\frac{27\gamma^2}{\alpha^2\delta^2}+\gamma^2+\tau_0^2\left(\frac{4\beta^2}{\kappa^2}+\frac{2L^2}{\pi^2\kappa^2}\right)+\frac{2\tau_0^2\beta^2L^2}{\pi^2\alpha\kappa^2}\right\}^{-1} \equiv \frac{\gamma\beta}{2}J_1,$$

$$\varepsilon \leq \frac{\gamma\beta}{2}\left\{\frac{2}{\kappa^2}\left(\frac{27}{\delta^2}+\frac{3\beta}{\alpha\delta}+\frac{3\beta^2}{4\alpha^2}+1\right)+\frac{4\beta^2}{\kappa^2}+\frac{2L^2}{\pi^2\kappa^2}+\frac{2\beta^2L^2}{\pi^2\alpha\kappa^2}\right\}^{-1} \equiv \frac{\gamma\beta}{2}J_2,$$

i.e.

$$\varepsilon \leq \varepsilon_1 := \min\left\{\frac{\pi^2\alpha}{2L^2}, \frac{1}{8\alpha^2}, \frac{\pi^2}{(L^2+\pi^2\delta^2)\alpha}, \frac{\gamma\beta J_1}{2}, \frac{\gamma\beta J_2}{2}\right\}. \tag{2.24}$$

Choosing ε as in (2.24) we obtain from (2.23)

$$\frac{d}{dt}F(t) \leq -d_1\int_0^L q^2 + q_t^2 + u_{tx}^2 + \theta_t^2 + u_{tt}^2 + u_t^2 + \theta^2 + u_x^2 dx \tag{2.25}$$

with

$$d_1 := \min\left\{\frac{\varepsilon}{2}, \frac{1}{4}, \frac{1}{\alpha}, \frac{1}{J_1}, \frac{1}{J_2}\right\}, \tag{2.26}$$

which implies

$$\frac{d}{dt}F(t) \leq -d_2\underbrace{(E_1(t)+E_2(t)}_{=:E(t)}, \tag{2.27}$$

where

$$d_2 := \frac{d_1}{2}\min\{\kappa\delta, \kappa\delta\alpha, \kappa\beta, \beta\gamma\tau_0\}. \tag{2.28}$$

On the other hand we have

$$\exists \varepsilon_2 > 0\ \ \exists C_1, C_2 > 0\ \ \forall \varepsilon \leq \varepsilon_2\ \ \forall t \geq 0: \quad C_1E(t) \leq F(t) \leq C_2E(t), \tag{2.29}$$

where C_1, C_2 are determined as follows: Let

$$H(t) := F(t) - \frac{1}{\varepsilon}E(t),$$

then

$$|H(t)| \leq C_1E(t)$$

with

$$C_1 := \max\left\{\left(\frac{3\beta\gamma}{\alpha^2\delta\kappa}+\frac{3\gamma}{2\alpha^2\delta}+\frac{3\beta\tau_0}{2\alpha^2\delta\kappa}+\frac{3}{2\kappa\delta\alpha}+\frac{\gamma}{2}+\frac{1}{\kappa^2}\right)/\left(\frac{\beta\tau_0}{2}\right),\right.$$
$$\left.\frac{1}{\alpha^2\kappa\delta},\frac{3\gamma}{\alpha^2\kappa\delta^2},\left(\frac{3\beta\tau_0}{2\alpha^2\delta\kappa}+\frac{3\tau_0\gamma}{2\kappa\alpha\delta}+\frac{\gamma\tau_0^2}{\kappa^2}\right)/\left(\frac{\beta\tau_0}{2}\right),\frac{3\tau_0\gamma+3}{\alpha\delta^2\kappa^2}\right\}. \tag{2.30}$$

Choosing

$$\varepsilon \le \varepsilon_2 := \frac{1}{2C_1} \tag{2.31}$$

and

$$C_2 := \frac{1}{\varepsilon}+C_1, \tag{2.32}$$

finally

$$\varepsilon := \min\{\varepsilon_1,\varepsilon_2\}, \tag{2.33}$$

we have fixed ε and the validity of (2.29) which implies together with (2.27)

$$\frac{d}{dt}F(t) \le -d_0F(t) \tag{2.34}$$

with

$$d_0 := \frac{d_2}{C_2}, \tag{2.35}$$

hence

$$F(t) \le e^{-d_0t}F(0).$$

Applying (2.29) again we have proved

$$E(t) \le C_0e^{-d_0t}E(0) \tag{2.36}$$

with

$$C_0 := \frac{C_2}{C_1}, \tag{2.37}$$

and it holds

Theorem 2.1 *Let (u,θ,q) be the solution to (2.1)– (2.5). Then the associated energy of first and second order*

$$E(t) = E_1(t)+E_2(t) = \frac{1}{2}\sum_{j=1}^{2}\int_0^L \kappa\delta(\partial_t^{j-1}u_t)^2+\kappa\delta\alpha(\partial_t^{j-1}u_x)^2+\kappa\beta\left(\partial_t^{j-1}\overline{\theta}\right)^2$$
$$+\gamma\beta\tau_0(\partial_t^{j-1}q)^2\Big\}(t,x)dx$$

decays exponentially, i.e.

$$\exists d_0, C_0>0\ \ \forall t\ge 0: \quad E(t)\le C_0e^{-d_0t}E(0).$$

Bounds for d_0 and C_0 can be given explicitly in terms of the coefficients $\alpha,\beta,\gamma,\delta,\tau_0$ and L.

In conclusion we have seen that the qualitative behavior of system (2.1)–(2.5) is the same as that for the associated classical system with $\tau_0 = 0$ and the boundary condition

$$u(t,0) = u(t,L) = \theta_x(t,0) = \theta_x(t,L) = 0, \qquad t \geq 0. \tag{2.38}$$

In section 5 we shall shed another light on the comparison of the systems as $\tau_0 \to 0$. Here we now compare the decay estimates for a real material.

For isotropic silicon and a medium temperature of 300 K we have the following approximate values for the material coefficients

$$\begin{aligned} &\alpha = 96,22 \cdot 10^6 \left[\frac{m^2}{s^2}\right], \quad \beta = 391,62 \left[\frac{m^2}{s^2 K}\right], \quad \gamma = 5 \cdot 99 \cdot 10^{-7} \left[\frac{s^2 K}{kg}\right], \\ &\delta = 163,82[K], \quad \kappa = 148 \left[\frac{W}{mK}\right] \quad \tau_0 = 10^{-12}[s], \quad L = 6 \cdot 25 \cdot 10^{-6}[m]. \end{aligned} \tag{2.39}$$

Successively we can approximately compute $\varepsilon_1, C_1, \varepsilon_2, \varepsilon, C_2, d_1, d_2, C_0$ and d_0 from (2.24), (2.30), (2.31), (2.33), (2.32), (2.26), (2.28), (2.37) and (2.35), respectively, getting finally

$$d_0 \approx 1,07 \cdot 10^{-56}.$$

In particular, we can get

$$d_0 = O(\tau_0^3) \text{ as } \tau_0 \to 0. \tag{2.40}$$

Although the estimate for d_0 is very coarse and might be far from being sharp, it indicates a slow decay of the energy in usually measured time periods. The relation (2.40) of course does not imply that solutions to the limiting case $\tau_0 = 0$ do not decay. Instead one can compute a better decay rate as follows. Consider the system (2.1)–(2.5) corresponding to $\tau_0 = 0$, i.e.

$$\tilde{u}_{tt} - \alpha \tilde{u}_{xx} + \beta \tilde{\theta}_x = 0, \tag{2.41}$$

$$\tilde{\theta}_t - \gamma\kappa \tilde{\theta}_{xx} + \delta \tilde{u}_{tx} = 0, \tag{2.42}$$

$$\tilde{u}(0,\cdot) = \tilde{u}_0, \quad \tilde{u}_t(0,\cdot) = \tilde{u}_1, \quad \tilde{\theta}(0,\cdot) = \tilde{\theta}_0, \tag{2.43}$$

$$\tilde{u}(t,0) = \tilde{u}(t,L) = \tilde{\theta}_x(t,0) = \tilde{\theta}_x(t,L) = 0. \tag{2.44}$$

For (2.41)–(2.44) the exponential decay of the first energy

$$\tilde{E}_1(t) := \frac{1}{2} \int_0^L (\delta \tilde{u}_t^2 + \delta\alpha \tilde{u}_x^2 + \beta \tilde{\theta}^2), (t,x) dx$$

where we assume

$$\int_0^L \tilde{\theta}_0 dx = 0.$$

We repeat the calculations that allow to even estimate $\tilde{E}_1$ directly, second-order derivatives are not involved.

We have

$$\frac{d}{dt}\tilde{E}_1 = -\gamma\kappa\beta \int_0^L \tilde{\theta}_x^2 dx. \tag{2.45}$$

Computing

$$\begin{aligned}
&\frac{d}{dt}\left(\int_0^L \int_0^x \tilde{\theta}(t,y)dy\, \tilde{u}_t(t,x)dx\right) \\
&\quad = \int_0^L (\gamma\kappa\tilde{\theta}_x - \delta\tilde{u}_t)\tilde{u}_t dx - \int_0^L \tilde{\theta}\alpha\tilde{u}_x dx + \int_0^L \beta\tilde{\theta}^2 dx \\
&\quad \le -\frac{3\delta}{4}\int_0^L \tilde{u}_t^2 dx + \frac{\alpha\delta}{16}\int_0^L \tilde{u}_x^2 dx + \left(\frac{\kappa^2\gamma^2}{\delta} + \Big(\frac{4\alpha}{\delta} + \beta\Big)\frac{L^2}{\pi^2}\right)\int_0^L \tilde{\theta}_x^2 dx,
\end{aligned} \tag{2.46}$$

and

$$\begin{aligned}
\frac{d}{dt}\frac{\delta}{4}\int_0^L \tilde{u}_t\tilde{u}\, dx &= \frac{\delta}{4}\int_0^L (\alpha\tilde{u}_{xx} - \beta\tilde{\theta}_x)\tilde{u}\, dx + \frac{\delta}{4}\int_0^L \tilde{u}_t^2 dx \\
&\le -\frac{3\alpha\delta}{16}\int_0^L \tilde{u}_x^2\, dx + \frac{4\beta^2\delta L^2}{\pi^2\alpha}\int_0^L \tilde{\theta}_x^2 dx + \frac{\delta}{4}\int_0^L \tilde{u}_t^2 dx.
\end{aligned} \tag{2.47}$$

Combining (2.45)–(2.47) we get for $N > 0$ large and

$$\tilde{F}(t) := N\tilde{E}_1(t) + \int_0^L \int_0^x \tilde{\theta}(t,y)\, dy\, \tilde{u}_t(t,x)dx + \frac{\delta}{4}\int_0^L \tilde{u}_t(t,x)\tilde{u}(t,x)dx, \tag{2.48}$$

$$\begin{aligned}
\frac{d}{dt}\tilde{F}(t) &\le -\tilde{c}_1 \int_0^L (\tilde{u}_t^2 + \tilde{u}_x^2 + \tilde{\theta}^2)(t,x)dx \\
&\le -\frac{\tilde{c}_1}{\tilde{c}_2}\cdot \tilde{E}_1(t),
\end{aligned} \tag{2.49}$$

where

$$\tilde{c}_1 := \min\left\{\frac{\pi^2}{L^2}\Big[N\gamma\kappa\beta - \Big(\frac{\kappa^2\gamma^2}{\delta} + \frac{(4\alpha+\beta\delta)L^2}{\delta\pi^2} + \frac{4\beta^2\delta L^2}{\alpha\pi^2}\Big)\Big], \frac{\delta}{2}, \frac{\alpha\delta}{8}\right\}, \tag{2.50}$$

$$\tilde{c}_2 := \frac{1}{2}\max\{\delta, \alpha\delta, \beta\}, \tag{2.51}$$

Choosing N such that

$$N\gamma\kappa\beta = 2\Big(\frac{\kappa^2\gamma^2}{\delta} + \frac{(4\alpha+\beta\delta)L^2}{\delta\pi^2} + \frac{4\beta^2\delta L^2}{\alpha\pi^2}\Big),$$

and observing

$$\tilde{F}(t) \leq \tilde{c}_3 E_1(t)$$

with

$$\tilde{c}_3 := N + \max\Big\{\frac{4+\delta}{4\delta}, \frac{L^2}{4\pi^2\alpha}, \frac{1}{\beta L^2}\Big\} \tag{2.52}$$

we conclude for some constant $\tilde{C}_0 > 0$:

$$\tilde{E}_1(t) \leq \tilde{C}_0 e^{-\tilde{d}_0 t} \tilde{E}_1(0)$$

with

$$\tilde{d}_0 := \frac{\tilde{c}_1}{\tilde{c}_2 \tilde{c}_3}. \tag{2.53}$$

Computing an approximate value for $\tilde{d}_0$ from (2.50) (2.53), we get

$$\tilde{d}_0 \approx 6,66 \cdot 10^{-20}$$

which is significantly larger than d_0 above, but still indicates a slow decay "in the beginning".

We remark that now it was possible to just treat the first energy, while the second-order energy in classical thermoelasticity ($\tau_0 = 0$) only is involved for more complicated boundary conditions like $u = \theta = 0$, cp. [16], [10]. For the proof of Theorem 2.1 it seems to be necessary to use second-order derivatives because of the more complicated system for $\tau_0 \neq 0$.

3 Rigidly clamped and constant temperature

Next we consider the differential equations (2.1)–(2.3), the initial conditions (2.4), but now with the boundary conditons

$$u(t,0) = u(t,L) = \theta(t,0) = \theta(t,L) = 0, \quad t \geq 0, \tag{3.1}$$

modeling a situation where the elastic body is rigidly clamped and held at constant temperature on the boundary.

Keeping the notation from section 2, we first get similar to (2.8), (2.9), (2.12)

$$\frac{d}{dt}E_1(t) = -\beta\gamma \int_0^L q^2 dx, \tag{3.2}$$

$$\frac{d}{dt}E_2(t) = -\beta\gamma \int_0^L q_t^2 dx, \tag{3.3}$$

$$\frac{2}{3}\int_0^L u_{xx}^2 dx + \frac{1}{\alpha}\frac{d}{dt}\left(\int_0^L u_{tx}u_x dx\right) \leq \frac{1}{\alpha}\int_0^L u_{tx}^2 dx + \frac{3\beta^2}{4\alpha^2}\int_0^L \theta_x^2 dx. \tag{3.4}$$

Trying to carry over the estimate (2.13) we end up with an additional term — a boundary term — and we get

$$\begin{aligned}
&\frac{2}{\alpha}\int_0^L u_{tx}^2 + \frac{1}{2}\int_0^L u_{xx}^2 dx + \frac{d}{dt}\Big\{\int_0^L \frac{1}{\alpha}u_{tx}u_x - \frac{3\gamma}{\alpha^2\delta}qu_{tt} + \frac{3\beta\tau_0\gamma}{\alpha^2\delta\kappa}qq_t \\
&\qquad + \frac{3\beta\gamma}{\alpha^2\delta\kappa}q^2 + \frac{3\tau_0}{\alpha\delta\kappa}q_t u_t + \frac{3}{\kappa\alpha\delta}qu_t dx\Big\} \\
&\leq \frac{27\gamma^2}{\alpha^2\delta^2}\int_0^L q_t^2 dx + \Big(\frac{27}{\delta^2} + \frac{3\beta}{\alpha\delta} + \frac{3\beta^2}{4\alpha^2}\Big)\int_0^L \theta_x^2 dx - \frac{3\gamma}{\alpha\delta}\Big[qu_{tx}\Big]_{x=0}^{x=L}.
\end{aligned} \tag{3.5}$$

(2.16) holds analogously,

$$\int_0^L u_{tt}^2 + u_t^2 + \theta^2 dx \leq 2\alpha^2\int_0^L u_{xx}^2 + \Big(2\beta^2 + \frac{L^2}{\pi^2}\Big)\frac{2}{\kappa^2}\int_0^L \tau_0^2 q_t^2 + q^2 dx + \frac{L^2}{\pi^2}\int_0^L u_{tx}^2 dx, \tag{3.6}$$

also (2.17) carries over,

$$\frac{\alpha}{2}\int_0^L u_x^2 dx \leq \frac{L^2}{\pi^2\alpha}\int_0^L u_{tt}^2 dx + \frac{\beta^2 L^2}{\pi^2\alpha}\int_0^L \theta x^2 dx, \tag{3.7}$$

as well as (2.18),

$$\int_0^L \theta_t^2 dx - \frac{d}{dt}\left(\int_0^L 2\gamma q\theta_x dx\right) \leq \gamma^2\int_0^L q_t^2 dx + \delta^2\int_0^L u_{tx}^2 dx + \int_0^L \theta_x^2 dx. \tag{3.8}$$

The boundary term is estimated as follows, cp. [16], [10],

$$\left|\Big[\frac{3\gamma}{\alpha\delta}qu_{tx}\Big]_{x=0}^{x=L}\right| \leq \frac{9\gamma^2}{4\alpha^2\delta^2\hat\varepsilon}\Big(|q(L)|^2 + |q(0)|^2\Big) + \hat\varepsilon\Big(|u_{tx}(L)|^2 + |u_{tx}(0)|^2\Big), \tag{3.9}$$

for some $\hat\varepsilon > 0$,

$$\begin{aligned}
|q(L)|^2 + |q(0)|^2 &\leq 2\Big(1 + \frac{L}{\hat\varepsilon^2}\Big)\int_0^L q^2 dx + 2\hat\varepsilon^2\int_0^L q_x^2 dx \\
&\leq 2\Big(1 + \frac{L}{\hat\varepsilon^2}\Big)\int_0^L q^2 dx + \frac{4\hat\varepsilon^2}{\gamma^2}\int_0^L \theta_t^2 dx + \frac{4\hat\varepsilon^2\delta^2}{\gamma^2}\int_0^L u_{tx}^2 dx.
\end{aligned} \tag{3.10}$$

Combining (3.9) and (3.10) we get

$$\begin{aligned}
\left|\Big[\frac{3\gamma}{\alpha\delta}qu_{tx}\Big]_{x=0}^{x=L}\right| &\leq \frac{9\gamma^2(\hat\varepsilon^2 + L)}{2\alpha^2\delta^2\hat\varepsilon^3}\int_0^L q^2 dx \\
&\quad + \frac{9\hat\varepsilon}{\alpha^2\delta^2}\int_0^L \theta_t^2 dx + \frac{9\hat\varepsilon}{\alpha^2}\int_0^L u_{tx}^2 dx \\
&\quad + \hat\varepsilon\Big(|u_{tx}(L)|^2 + |u_{tx}(0)|^2\Big).
\end{aligned} \tag{3.11}$$

Differentiating (2.1) with respect to t and multiplying by $\varphi(x)u_{tx}(t,x)$, where

$$\varphi(x) := L - 2x, \tag{3.12}$$

we obtain

$$\frac{d}{dt}\left(\int_0^L u_{tt}\varphi u_{tx}dx\right) - \int_0^L u_{tt}\varphi u_{ttx}dx - \alpha\int_0^L \varphi u_{txx}u_{tx}dx + \beta\int_0^L \theta_{tx}\varphi u_{tx} = 0,$$

whence

$$\frac{d}{dt}\left(\int_0^L u_{tt}\varphi u_{tx}dx\right) + \frac{L}{2}\Big(u_{tt}^2(L) + u_{tt}^2(0)\Big) - \int_0^L u_{tt}^2dx$$
$$+\frac{L\alpha}{2}\Big(u_{tx}^2(L) + u_{tx}^2(0)\Big) - \alpha\int_0^L u_{tx}^2dx + \beta\int_0^L \theta_{tx}\varphi u_{tx}dx = 0 \tag{3.13}$$

Multiplication of (2.2) by $-\frac{\beta}{\delta}\varphi\theta_{tx}$ yields

$$-\frac{\beta}{\delta}\int_0^L \theta_t\varphi\theta_{tx}dx - \frac{\beta\gamma}{\delta}\int_0^L q_x\varphi\theta_{tx}dx - \beta\int_0^L u_{tx}\varphi\theta_{tx} = 0,$$

whence

$$\frac{\beta L}{2\delta}\Big(\theta_t^2(L) + \theta_t^2(0)\Big) - \frac{\beta}{\delta}\int_0^L \theta_t^2dx - \frac{\beta\gamma}{d}\frac{d}{dt}\left(\int_0^L q_x\varphi\theta_x dx\right) + \frac{\beta\gamma}{\delta}\int_0^L q_{tx}\varphi\theta_x dx$$
$$-\beta\int_0^L u_{tx}\varphi\theta_{tx}dx = 0,$$

which implies, using (2.3),

$$\frac{\beta L}{2\delta}\Big(\theta_t^2(L) + \theta_t^2(0)\Big) - \frac{\beta}{\delta}\int_0^L \theta_t^2dx - \frac{\beta\gamma}{\delta}\frac{d}{dt}\left(\int_0^L q_x\varphi\theta_x dx\right) - \frac{\beta\gamma}{\delta\tau_0}\int_0^L q_x\varphi\theta_x dx$$
$$+\frac{\gamma\beta\kappa L}{2\delta\tau_0}\Big(\theta_x^2(L) + \theta_x^2(0)\Big) - \frac{\beta\gamma\kappa}{\delta\tau_0}\int_0^L \theta_x^2dx - \beta\int_0^L u_{tx}\varphi\theta_{tx}dx = 0. \tag{3.14}$$

Combining (3.13), (3.14) we conclude

$$\frac{d}{dt}\left(\int_0^L u_{tt}\varphi u_{tx} - \frac{\beta\gamma}{\delta}q_x\varphi\theta_x\right)dx + \frac{\alpha L}{2}(u_{tx}^2(L) + u_{tx}^2(0))$$
$$\leq \int_0^L u_{tt}^2dx + \alpha\int_0^L u_{tx}^2dx + \frac{\beta}{\delta}\int_0^L \theta_t^2dx + \frac{\beta\gamma\kappa}{\delta\tau_0}\int_0^L \theta_x^2dx$$
$$+\frac{\beta^2\gamma^2}{4\delta^2\tau_0^2}\int_0^L \theta_x^2dx + L^2\int_0^L q_x^2dx$$

whence, using (2.2),

$$
\begin{aligned}
&\frac{d}{dt}\left(\frac{2\hat\varepsilon}{\alpha L}\int_0^L u_{tt}\varphi u_{tx} - \frac{\beta\gamma}{\delta} q_x\varphi\theta_x dx\right) + \hat\varepsilon(u_{tx}^2(L) + u_{tx}^2(0)) \\
&\quad\leq \frac{2\hat\varepsilon}{\alpha L}\int_0^L u_{tt}^2 dx + \left(\frac{2\hat\varepsilon}{L} + \frac{4\delta^2\hat\varepsilon}{\alpha L\gamma^2}\right)\int_0^L u_{tx}^2 dx \\
&\qquad + \left(\frac{4\hat\varepsilon}{\gamma^2\alpha L} + \frac{2\beta\hat\varepsilon}{\alpha L\delta}\right)\int_0^L \theta_t^2 dx + \left(\frac{2\beta\gamma\kappa\hat\varepsilon}{\alpha L\delta\tau_0} + \frac{\beta^2\gamma^2\hat\varepsilon}{2\alpha L\delta^2\tau_0^2}\right)\int_0^L \theta_x^2 dx
\end{aligned}
\tag{3.15}
$$

follows.

With (3.11) and (3.15) we can estimate

$$
\begin{aligned}
\left|\left[\frac{3\gamma}{\alpha\delta} q u_{tx}\right]_{x=0}^{x=L}\right| \leq\ & \frac{9\gamma^2(\hat\varepsilon^2 + L)}{2\alpha^2\delta^2\hat\varepsilon^3}\int_0^L q^2 dx + \hat\varepsilon\left(\frac{9}{\alpha^2\delta^2} + \frac{2\beta}{\alpha L\delta} + \frac{4}{\gamma^2\alpha L}\right)\int_0^L \theta_t^2 dx \\
&+\hat\varepsilon\left(\frac{9}{\alpha^2} + \frac{2}{L} + \frac{4\delta^2}{\alpha L\gamma^2}\right)\int_0^L u_{tx}^2 dx + \frac{2\hat\varepsilon}{\alpha L}\int_0^L u_{tt}^2 dx \\
&+\hat\varepsilon\left(\frac{2\beta\gamma\kappa}{\alpha L\delta\tau_0} + \frac{\beta^2\gamma^2}{2\alpha L\delta^2\tau_0^2}\right)\int_0^L \theta_x^2 dx \\
&-\frac{d}{dt}\left(\frac{2\hat\varepsilon}{\alpha L}\int_0^L u_{tt}\varphi u_{tx} - \frac{\beta\gamma}{\delta} q_x\varphi\theta_x dx\right).
\end{aligned}
\tag{3.16}
$$

Multiplying both sides of (3.8) by $\frac{1}{4\alpha\delta^2}$ and combining the result with (3.5) and (3.16), we obtain for sufficiently small $\hat\varepsilon$ the estimate

$$
\begin{aligned}
&\frac{1}{\alpha}\int_0^L u_{tx}^2 dx + \frac{1}{4}\int_0^L u_{xx}^2 + \frac{1}{8\alpha\delta^2}\int_0^L \theta_t^2 + \frac{d}{dt}G(t) \\
&\quad\leq c_1\int_0^L q_t + q^2 dx,
\end{aligned}
\tag{3.17}
$$

where

$$
\begin{aligned}
G(t) := & \int_0^L \frac{1}{\alpha}u_{tx}u_x - \frac{3\gamma}{\alpha^2\delta} q u_{tt} + \frac{3\beta\tau_0\gamma}{\alpha^2\delta\kappa} q q_t + \frac{3\beta\gamma}{\alpha^2\delta\kappa} q^2 + \frac{3\tau_0}{\alpha\delta\kappa} q_t u_t \\
&+\frac{3}{\kappa\alpha\delta} q u_t + \frac{2\hat\varepsilon}{\alpha L}(u_{tt}\varphi u_{tx} - \frac{\beta\gamma}{\delta} q_x\varphi\theta_x) - \frac{\gamma}{2\alpha\delta^2} q\theta_x dx,
\end{aligned}
$$

and

$$c_1 = c_1(\hat\varepsilon, \alpha, \beta, \gamma, \delta, \tau_0, L)$$

can be given explicitly.

Now we can proceed as in section 2 and define

$$F(t) := \frac{1}{\varepsilon}(E_1(t) + E_2(t)) + G(t),$$

$\varepsilon > 0$ small enough, as Lyapunov function to derive

Theorem 3.1 *Let (u, θ, q) be the solution to (2.1)–(2.4), (3.1). Then the associated energy of first and second order,*

$$\begin{aligned} E(t) &= E_1(t) + E_2(t) = \frac{1}{2}\sum_{j=1}^{2}\int_0^L \kappa\delta(\partial_t^{j-1}u_t)^2 + \kappa\delta\alpha(\partial_t^{j-1}u_x)^2 + \kappa\beta\left(\partial_t^{j-1}\theta\right)^2 \\ &\quad + \gamma\beta\tau_0(\partial_t^{j-1}q)^2\Big\}(t,x)dx \end{aligned}$$

decays exponentially, i.e.

$$\exists d_0, C_0 > 0 \quad \forall t \geq 0 : \quad E(t) \leq C_0 e^{-d_0 t} E(0).$$

Bounds for d_0 and C_0 can be given explicitly in terms of the coefficients $\alpha, \beta, \gamma, \delta, \tau_0$ and L.

4 Mixed boundary conditions

Finally, we consider the system (2.1)–(2.3) with the boundary conditions

$$u(t, L) = \theta(t, L) = \alpha u_x(t, 0) - \beta\theta(t, 0) = \theta_x(t, 0) = 0, \quad t \geq 0, \tag{4.1}$$

which arise for instance in pulsed laser heating of solids, cp. [23]. Assuming

$$q_0(0) = 0, \tag{4.2}$$

we conclude from (2.3)

$$q(t, 0) = 0, \quad t \geq 0. \tag{4.3}$$

The set of boundary conditions (4.1), (4.3) leads to further difficulties in the proof of the result on exponential decay, which will be overcome by introducing further multipliers on the boundary now. Using the same notation as before, we get for the energy E_1 of first order, observing the boundary conditions (4.1), (4.3),

$$\frac{d}{dt}E_1(t) = -\beta\gamma\int_0^L q^2 dx, \tag{4.4}$$

similarly,

$$\frac{d}{dt}E_2(t) = -\beta\gamma\int_0^L q_t^2 dx. \tag{4.5}$$

Multiplying (2.1) by $\frac{1}{\alpha}u_{xx}$ we now obtain

$$\begin{aligned}
\int_0^L u_{xx}^2 dx &= -\frac{1}{\alpha}u_{tt}(0)u_x(0) - \frac{1}{\alpha}\int_0^L u_{ttx}u_x dx + \frac{\beta}{\alpha}\int_0^L \theta_x u_{xx} dx \\
&\leq -\frac{\beta}{\alpha^2}u_{tt}(0)\theta(0) - \frac{1}{\alpha}\frac{d}{dt}\left(\int_0^L u_{tx}u_x dx\right) + \frac{1}{\alpha}\int_0^L u_{tx}^2 dx + \frac{3\beta^2}{4\alpha^2}\int_0^L \theta_x^2 dx + \frac{1}{3}\int_0^L u_{xx}^2 dx
\end{aligned}$$

implying

$$\frac{2}{3}\int_0^L u_{xx}^2 dx + \frac{d}{dt}\left(\frac{1}{\alpha}\int_0^L u_{tx}u_x dx\right) \leq \frac{1}{\alpha}\int_0^L u_{tx}^2 dx + \frac{3\beta^2}{4\alpha^2}\int_0^L \theta_x^2 dx - \frac{\beta}{\alpha^2}u_{tt}(0)\theta(0). \tag{4.6}$$

Multiplying (2.2) by $\frac{3}{\alpha\delta}u_{tx}$ we conclude

$$\begin{aligned}
\frac{3}{\alpha}\int_0^L u_{tx}^2 dx &= -\frac{3\gamma}{\alpha\delta}q(L)u_{tx}(L) + \frac{d}{dt}\left(\frac{3\gamma}{\alpha^2\delta}\int_0^L q u_{tt}\right) + \frac{d}{dt}\left(\frac{3\gamma\beta}{\alpha^2\delta}\int_0^L q\theta_x dx\right) \\
&\quad -\frac{3\gamma}{\alpha\delta}\int_0^L q_t u_{xx} dx + \frac{3}{\alpha\delta}\theta_t(0)u_t(0) + \frac{d}{dt}\left(\frac{3}{\alpha\delta}\int_0^L \theta_x u_t dx\right) \\
&\quad -\frac{3}{\delta}\int_0^L \theta_x u_{xx} dx + \frac{3\beta}{\alpha\delta}\int_0^L \theta_x^2 dx \\
&\leq -\frac{3\gamma}{\alpha\delta}q(L)u_{tx}(L) + \frac{d}{dt}\left(\int_0^L \frac{3\gamma}{\alpha^2\delta}q u_{tt} - \frac{3\gamma\beta\tau_0}{\alpha^2\delta\kappa}q q_t - \frac{3\gamma\beta}{\alpha^2\delta\kappa}q^2 dx\right) \\
&\quad +\frac{1}{12}\int_0^L u_{xx}^2 dx + \frac{27\gamma^2}{\alpha^2\delta^2}\int_0^L q_t^2 dx + \frac{3}{\alpha\delta}\theta_t(0)u_t(0) + \frac{d}{dt}\left(\int_0^L \frac{3}{\alpha\delta}\theta_x u_t dx\right) \\
&\quad +\frac{1}{12}\int_0^L u_{xx}^2 dx + \left(\frac{27}{\delta^2} + \frac{3\beta}{\alpha\delta}\right)\int_0^L \theta_x^2 dx.
\end{aligned} \tag{4.7}$$

Combining (4.6) and (4.7) we obtain

$$\begin{aligned}
&\frac{2}{\alpha}\int_0^L u_{tx}^2 dx + \frac{1}{2}\int_0^L u_{xx}^2 dx + \frac{d}{dt}\left(\int_0^L \frac{1}{\alpha}u_{tx}u_x - \frac{3\gamma}{\alpha^2\delta}q u_{tt} + \frac{3\gamma\beta\tau_0}{\alpha^2\delta\kappa}q q_t\right. \\
&\qquad \left. + \frac{3\gamma\beta}{\alpha^2\delta\kappa}q^2 + \frac{3\tau_0}{\alpha\delta\kappa}q_t u_t + \frac{3}{\alpha\delta\kappa}q u_t dx\right) \\
&\leq c_1\int_0^L q_t^2 dx + c_2\int_0^L \theta_x^2 dx \\
&\qquad -\frac{\beta}{\alpha^2}u_{tt}(0)\theta(0) + \frac{3}{\alpha\delta}\theta_t(0)u_t(0) - \frac{3\gamma}{\alpha\delta}q(L)u_{tx}(L),
\end{aligned} \tag{4.8}$$

where $c_1, c_2, \ldots$ will denote constants only depending on the coefficients (or on $\hat{\varepsilon}$ below), possibly denoting different values in different places, and in principle easy to determinate as in section 2 demonstrated.

Again we obtain using the first Poincaré estimate for u_t and θ, and (2.1) for u_{tt},

$$\int_0^L u_{tt}^2 + u_t^2 + \theta^2 dx \leq 2\alpha^2 \int_0^L u_{xx}^2 dx + c_1 \int_0^L q_t^2 + q^2 dx + \frac{L^2}{\pi^2} \int_0^L u_{tx}^2 dx. \tag{4.9}$$

Multiplying (2.1) by u we get again (cp. (2.17))

$$\frac{\alpha}{2} \int_0^L u_x^2 dx \leq c_1 \int_0^L (u_{tt}^2 + \theta_x^2) dx. \tag{4.10}$$

Multiplication of (2.2) by θ_t yields as in (2.18), for $\mu > 0$,

$$\mu \int_0^L \theta_t^2 dx - \frac{d}{dt}\mu \int_0^L 2\gamma q \theta_x dx \leq \mu\gamma^2 \int_0^L q_t^2 dx + \mu\delta^2 \int_0^L u_{tx}^2 dx + \mu \int_0^L \theta_x^2 dx. \tag{4.11}$$

The three boundary terms arising in (4.8) are dealt with as follows. As in (3.11) we have, for $\hat{\varepsilon} > 0$,

$$\begin{aligned} |\frac{3\gamma}{\alpha\delta} q(L) u_{tx}(L) \quad &\leq \quad \frac{9\gamma^2(\hat{\varepsilon}^2 + L)}{2\alpha^2\delta^2\hat{\varepsilon}^3} \int_0^L q^2 dx + \frac{9\hat{\varepsilon}}{\alpha^2\delta^2} \int_0^L \theta_t^2 dx \\ &\quad + \frac{9\hat{\varepsilon}^2}{\alpha^2} \int_0^L u_{tx}^2 dx + \hat{\varepsilon}|u_{tx}(L)|^2. \end{aligned} \tag{4.12}$$

Since (3.13) and (3.14) carry over, we have

$$\begin{aligned} &\frac{d}{dt}\left(\int_0^L u_{tt}\varphi u_{tx} - \frac{\beta\gamma}{\delta} q_x \varphi \theta_x dx \right) + \frac{L}{2} u_{tt}^2(0) + \frac{\alpha L}{2}(u_{tx}^2(L) + u_{tx}^2(0)) \\ &\quad + \frac{\beta L}{2\delta}\theta_t^2(0) + \frac{\gamma\beta\kappa L}{2\delta\tau_0}\theta_x^2(L) \\ &\quad \leq \int_0^L u_{tt}^2 dx + \alpha \int_0^L u_{tx}^2 dx + \frac{\beta}{\delta}\int_0^L \theta_t^2 dx + \left(\frac{\beta\gamma\kappa}{\delta\tau_0} + \frac{\beta^2\gamma^2}{4\delta^2\tau_0^2} \right) \int_0^L \theta_x^2 dx + L^2 \int_0^L q_x^2 dx, \end{aligned}$$

hence

$$\begin{aligned} &\frac{d}{dt}\left(\frac{2\hat{\varepsilon}}{\alpha L} \int_0^L u_{tt}\varphi u_{tx} - \frac{\beta\gamma}{\delta} q_x \varphi \theta_x dx \right) + \frac{\hat{\varepsilon}}{\alpha} u_{tt}^2(0) + \hat{\varepsilon}(u_{tx}^2(L) + u_{tx}^2(0)) + \frac{\beta\hat{\varepsilon}}{\alpha\delta}\theta_t^2(0) \\ &\quad \leq \hat{\varepsilon} c_1 \int_0^L u_{tt}^2 + u_{tx}^2 + \theta_t^2 + \theta_x^2 dx. \end{aligned} \tag{4.13}$$

The first boundary term in (4.8) is estimated as

$$\left| \frac{\beta}{\alpha^2} u_{tt}(0)\theta(0) \right| \leq \frac{\hat{\varepsilon}}{2\alpha} u_{tt}^2(0) + \frac{c_1}{\hat{\varepsilon}} \int_0^L q_t^2 + q^2 dx, \tag{4.14}$$

the remaining boundary term as

$$\begin{aligned}
\frac{3}{\alpha\delta}\theta_t(0)u_t(0) &= \frac{3}{\alpha\delta}\frac{d}{dt}(\theta(t,0)u_t(t,0)) - \frac{3}{\alpha\delta}\theta(0)u_{tt}(0) \\
&\le \frac{3}{\alpha\delta}\frac{d}{dt}(\theta(t,0)u_t(t,0)) + \frac{\hat\varepsilon}{2\alpha}u_{tt}^2(0) + \frac{c_1}{\hat\varepsilon}\int_0^L q_t^2 + q^2 dx. \qquad (4.15)
\end{aligned}$$

Summarizing (4.8) and (4.11)–(4.15), we obtain for $\hat\varepsilon, \mu$ sufficiently small,

$$\frac{1}{\alpha}\int_0^L u_{tx}^2 dx + \frac{1}{2}\int_0^L u_{xx}^2 dx + \mu\int_0^L \theta_t^2 dx + \frac{d}{dt}W(t) \le c_1\int_0^L (q_t^2 + q^2)dx,$$

where

$$\begin{aligned}
W(t) &:= \int_0^L \frac{1}{\alpha}u_{tx}u_x - \frac{3\gamma}{\alpha^2\delta}qu_{tt} + \frac{3\gamma\beta\tau_0}{\alpha^2\delta\kappa}qq_t + \frac{3\gamma\beta}{\alpha^2\delta\kappa}q^2 + \frac{3\tau_0}{\alpha\delta\kappa}q_t u_t \\
&\quad + \frac{3}{\alpha\delta\kappa}qu_t + \frac{2\hat\varepsilon}{\alpha L}(u_{tt}\varphi u_{tx} - \frac{\beta\gamma}{\delta}q_x\varphi\theta_x) - 2\mu\gamma q\theta_x dx \\
&\quad - \frac{3}{\alpha\delta}\theta(t,0)u_t(t,0).
\end{aligned}$$

Proceeding now as in section 2 for the Lyapunov function

$$F(t) := \frac{1}{\varepsilon}(E_1(t) + E_2(t)) + W(t),$$

we conclude

$$\frac{d}{dt}F(t) \le -c_1 E(t)$$

and, finally,

Theorem 4.1 *Let (u, θ, q) be the solution to (2.1) –(2.4), (4.1), (4.2). Then the associated energy of first and second order,*

$$\begin{aligned}
E(t) &= E_1(t) + E_2(t) = \frac{1}{2}\sum_{j=1}^{2}\int_0^L \kappa\delta(\partial_t^{j-1}u_t)^2 + \kappa\delta\alpha(\partial_t^{j-1}u_x)^2 + \kappa\beta(\partial_t^{j-1}\theta)^2 \\
&\quad + \gamma\beta\tau_0(\partial_t^{j-1}q)^2\}(t,x)dx
\end{aligned}$$

decays exponentially, i.e.

$$\exists d_0, C_0 > 0 \quad \forall t \ge 0: \qquad E(t) \le C_0 e^{-d_0 t}E(0).$$

Bounds for d_0 and C_0 can be given explicitly in terms of the coefficients $\alpha, \beta, \gamma, \delta, \tau_0$ and L.

5 The limit $\tau_0 \to 0$

We shall show that the energy of the difference of the solution $(u^{\tau_0}, \theta^{\tau_0}, q^{\tau_0})$ to (2.1)–(2.4), (3.1) and the solution $(\tilde{u}, \tilde{\theta}, \tilde{q})$ to the corresponding system with $\tau_0 = 0$ (compare section 2) vanishes of order τ_0^2 as $\tau_0 \to 0$, provided the values at $t = 0$ coincide. For this purpose let (v, ϱ, z) denote the difference, then (v, ϱ, z) satisfies

$$\begin{aligned} v_{tt} - \alpha v_{xx} + \beta \varrho_x &= 0, && (5.1)\\ \varrho_t + \gamma z_x + \delta v_{tx} &= 0, && (5.2)\\ \tau_0 z_t + z + \kappa \varrho_x &= \tau_0 \kappa \tilde{\theta}_{tx}, && (5.3) \end{aligned}$$

$$v(0, \cdot) = 0, \quad v_t(0, \cdot) = 0, \quad \varrho(0, \cdot) = 0, \quad z(0, \cdot) = 0, \tag{5.4}$$

$$v(t, 0) = v(t, L) = \varrho(t, 0) = \varrho(t, L) = 0. \tag{5.5}$$

Here we assumed the compatibility condition

$$q_0 = -\kappa \theta_{0,x}.$$

If E_1 denotes the energy of first order for (v, ϱ, z), i.e.

$$E_1(t) = \frac{1}{2} \int_0^L \{\kappa \delta v_t^2 + \kappa \delta \alpha v_x^2 + \kappa \beta \varrho^2 + \gamma \beta \tau_0 z^2\}(t, x) dx,$$

then we conclude

$$\begin{aligned} \frac{d}{dt} E_1(t) &= -\beta\gamma \int_0^L z^2 dx - \tau_0 \beta \gamma \kappa \int_0^L \tilde{\theta}_{tx} z \, dx \\ &\leq -\frac{\beta\gamma}{2} \int_0^L z^2 dx + \frac{\beta\gamma\tau_0^2\kappa^2}{2} \int_0^L \tilde{\theta}_{tx}^2 dx. \end{aligned}$$

Using (5.4) we obtain

$$E_1(t) \leq \tau_0^2 \frac{\beta\gamma\kappa^2}{2} \int_0^t \int_0^L \tilde{\theta}_{tx}^2(s, x) dx \, ds$$

from where we get for $T > 0$ fixed, $t \in [0, T]$:

$$\begin{aligned} E_1(t) &\leq \tau_0^2 \frac{\beta\gamma\kappa^2}{2} \int_0^T \int_0^L \tilde{\theta}_{tx}^2(s, x) dx \, ds \\ &= O(\tau_0^2), \text{ as } \tau_0 \to 0, && (5.6) \end{aligned}$$

also

$$\frac{E_1(t)}{\tau_0^2} \to 0 \text{ as } t \to 0. \tag{5.7}$$

Moreover, since

$$\int_0^\infty \int_0^L \tilde{\theta}_{tx}^2(s,x)dx\,ds < \infty$$

because of the exponential decay (cp. [16, 10]), the estimate (5.6) allows $T = \infty$, i.e. a uniform bound on the right-hand side,

$$\exists C > 0 \;\; \forall t \geq 0 : \quad E_1(t) \leq C\tau_0^2. \tag{5.8}$$

6 Local well-posedness

In this section we give a short derivation of the nonlinear equations in $\mathbb{R}^n$, $n = 1, 2, 3$, and present a local existence theorem for the nonlinear initial boundary value problem for $n = 1$. For the derivation compare [18, 1, 10], the latter two discussing the classical case with Fourier's law, see also [2, 21] for extensive reviews on generalized thermoelasticity.

For a body with undistorted reference configuration $\Omega \subset \mathbb{R}^n$ let $u(t,x) = X(t,x) - x$ denote the displacement vector, where X is the position vector. If $T = T(t,x)$ denotes the temperature, then, in absence of exterior forces, the balance law of linear momentum reads

$$\varrho u_{tt} - \operatorname{div} S = 0, \tag{6.1}$$

where ϱ is the material density and S is the Piola-Kirchhoff stress tensor. Henceforth taking densities like ϱ equal to one without loss of generality, the balance of energy in the absence of external heat supplies takes the form

$$\varepsilon_t - \operatorname{tr}\{SF_t\} + \operatorname{div} q = 0, \tag{6.2}$$

where ε denotes the internal energy, tr stands for the trace, F is the deformation gradient,

$$F = 1 + \nabla u,$$

and q is the heat flux. Denoting by η the specific entropy we get by

$$\psi := \varepsilon - T\eta$$

the free energy. The first constitutive assumption, namely that S, ε, η and q are functions of $(\nabla u, \theta, q, \nabla\theta)$ — we shall assume homogeneity, i.e. no extra dependence on x, throughout the paper —, is turned into the following relations after the second law of thermodynamics resp. the dissipation principle are applied:

$$\psi = \psi(\nabla u, \theta, q) \tag{6.3}$$

is independent of $\nabla\theta$, and

$$\eta = \eta(\nabla u, \theta, q) = -\psi_\theta(\nabla u, \theta, q), \quad S = S(\nabla u, \theta, q) = \psi_{\nabla u}(\nabla u, \theta, q). \tag{6.4}$$

For the heat flux Cattaneo's law is assumed, i.e.

$$\tau(\nabla u, \theta)q_t + q + k(\nabla u, \theta)\nabla\theta = 0, \tag{6.5}$$

where τ is the tensor of relaxation times and k is the symmetric heat conductivity tensor. Using the relations (6.3), (6.4) we obtain from (6.2)

$$(\theta + T_0)\{\tilde{a}(\nabla u, \theta, q)\theta_t - \operatorname{tr}[(S_\theta(\nabla u, \theta, q))' \nabla u_t]\} + \operatorname{div} q = \tilde{b}(\nabla u, \theta, q) q_t \tag{6.6}$$

with

$$\tilde{a} := -\psi_{\theta\theta}, \quad \tilde{b}(\nabla u, \theta, q) := (\theta + T_0)\psi_{\theta q}(\nabla u, \theta, q) - \psi_q(\nabla u, \theta, q).$$

The equations (6.1), (6.6), (6.5) are the differential equations modeling thermoelasticity with second sound. Further assumptions on the coefficent functions are given below.

Remark: There are theories that restrict the form of the free energy ψ, for the internal energy ε, for the entropy η, and for the stress tensor S essentially to a quadratic polynomial in q, e.g. for ψ of the type

$$\psi = \psi_0 + q \cdot Zq, \tag{6.7}$$

where ψ_0 and the tensor Z only depend on $(\nabla u, \theta)$, cp. [18, 22]. But this is not important for our results.

The system is completed by the initial conditions,

$$u(0, \cdot) = u_0, \quad u_t(0, \cdot) = u_1, \quad \theta(0, \cdot) = \theta_0, \quad q(0, \cdot) = q_0 \tag{6.8}$$

and the Dirichlet type boundary conditions for a rigidly clamped body held at constant temperature at the boundary,

$$u(t, \cdot) = 0, \quad \theta(t, \cdot) = 0, \qquad \text{on } \partial\Omega, \quad t \geq 0. \tag{6.9}$$

In one space dimension the bounded reference configuration is without loss of generality $\Omega = (0, 1)$, and the differential equations (6.1), (6.6), (6.5) can be written as

$$\begin{aligned} u_{tt} - a(u_x, \theta, q)u_{xx} + b(u_x, \theta, q)\theta_x &= r_1(u_x, \theta, q)q_x, && (6.10)\\ \theta_t + g(u_x, \theta, q)q_x + d(u_x, \theta, q)u_{tx} &= r_2(u_x, \theta, q)q_t, && (6.11)\\ \tau(u_x, \theta)q_t + q + k(u_x, \theta)\theta_x &= 0, && (6.12) \end{aligned}$$

where

$$a := S_{u_x}, \quad b := -S_\theta, \quad r_1 := S_q,$$

$$g := \frac{1}{(\theta + T_0)\tilde{a}}, \quad d := \frac{-S_\theta}{(\theta + T_0)\tilde{a}}, \quad r_2 := \frac{\tilde{b}}{(\theta + T_0)\tilde{a}}, \qquad (\text{ for } |\theta| < T_0).$$

The assumptions on $a, b, r_1, \tilde{a}$ (resp. g), d, r_2, τ, k are that it are smooth functions and that there exist positive constants c_1, c_2, c_3, c_4 and $K < T_0$ such that if $|u_x|$, $|\theta|$, $|q| \leq K$, then

$$a(u_x, \theta, q) \geq c_1, \quad \tau(u_x, \theta) \geq c_2, \quad k(u_x, \theta) \geq c_3, \quad \tilde{a}(u_x, \theta, q) \geq c_4$$

as well as

$$b(u_x, \theta, q) \neq 0 \neq d(u_x, \theta, q).$$

Moreover we shall assume

$$r_1 \equiv 0, \quad r_2 \equiv 0.$$

Observe that always $r_1(0,0,0) = r_2(0,0,0) = 0$ in accordance with the linearized equations, compare the previous sections. The coefficients $\alpha, \beta, \gamma, \delta, \tau_0, \kappa$ in the linearized equations are of course given by

$$\alpha = a(0,0,0), \quad \beta = b(0,0,0), \quad \gamma = g(0,0,0),$$
$$\delta = d(0,0,0), \quad \tau_0 = \tau(0,0), \quad \kappa = k(0,0).$$

For $s \geq [n/2] + 3 = 3$ being a fixed integer, we assume for the initial data

$$u_0 \in H^s(\Omega),\ u_1, \theta_0, q_0 \in H^{s-1}(\Omega), \quad \theta_0 \geq 0 \text{ in } \Omega, \tag{6.13}$$

and the compatibility conditions

$$u_m \in H^{s-m}(\Omega) \cap H_0^1(\Omega),\ \theta_{m-1} \in H^{s-m}(\Omega) \cap H_0^1(\Omega),\ u_s, \theta_{s-1} \in L^2(\Omega), \tag{6.14}$$

where m runs from 2 to $s-1$, and u_m and θ_{m-1} denote $\partial^m u(0,\cdot)/\partial t^m$ and $\partial^{m-1}\theta(0,\cdot)/\partial t^{m-1}$, respectively. Then we can formulate the following local existence theorem:

Theorem 6.1 *Let the assumptions above be satisfied. Then, for sufficiently small $T > 0$, the initial boundary value problem (6.8)–(6.12) has a unique solution (u, θ, q) with*

$$u \in \cap_{m=0}^{s-1} C^m([0,T], H^{s-m}(\Omega) \cap H_0^1(\Omega)), \quad \partial_t^s u \in C^0([0,T], L^2(\Omega)),$$
$$\theta, q \in \cap_{m=0}^{s-2} C^m([0,T], H^{s-1-m}(\Omega) \cap H_0^1(\Omega)), \quad \partial_t^{s-1}\theta, q \in C^0([0,T], L^2(\Omega)).$$

For the PROOF we just remark that it can be obtained along the lines of the proof of the similar result in hyperbolic-parabolic classical thermoelasticity, see chapter 5 in [10] or [3, 5]. Although we have now a purely hyperbolic system of second order for u with first-order parts for (θ, q), the techniques for the classical case carry over and are *a posteriori* justified through the *a priori* estimates that will be proved in section 7.

Since Tarabek in [22] is essentially in the situation of a Cauchy problem in one dimension, he can apply the results from [8] for his local existence result.

7 Global stability for the nonlinear system

Now we shall prove the global existence for the system of differential equations (6.10)–(6.12) together with the initial and boundary conditions (6.8) and (6.9), respectively. That is we look at the following initial boundary value problem:

$$u_{tt} - a(u_x, \theta, q)u_{xx} + b(u_x, \theta, q)\theta_x = 0, \tag{7.1}$$
$$\theta_t + g(u_x, \theta, q)q_x + d(u_x, \theta, q)u_{tx} = 0, \tag{7.2}$$
$$\tau(u_x, \theta)q_t + q + k(u_x, \theta)\theta_x = 0, \tag{7.3}$$

$$u(0,\cdot) = u_0, \quad u_t(0,\cdot) = u_1, \quad \theta(0,\cdot) = \theta_0, \quad q(0,\cdot) = q_0, \tag{7.4}$$
$$u(t,0) = u(t,1) = \theta(t,0) = \theta(t,1) = 0, \qquad t \geq 0. \tag{7.5}$$

Finally we assume for the initial conditions the regularity (6.13) and the compatibility conditions (6.14) for $s = 3$.

Under these assumptions, in particular those for the local existence theorem in section 6 (Theorem 6.1), we have

Theorem 7.1 *There exists a constant* $\varepsilon_0 > 0$ *such that if*

$$\Lambda_0 := \|u_0\|_{H^3}^2 + \|u_1\|_{H^2}^2 + \|\theta_0\|_{H^2}^2 + \|q_0\|_{H^2}^2 < \varepsilon_0$$

then (7.1)–(7.5) has a unique global solution

$$u \in \cap_{m=0}^3 C^m([0,\infty), H^{3-m}(\Omega)), \quad \theta, q \in \cap_{m=0}^2 C^m([0,\infty), H^{2-m}(\Omega));$$

moreover, the system is exponentially stable, that is, there exist constants $d_1, d_2 > 0$ *such that for* $t \geq 0$

$$\Lambda(t) := \sum_{j=0}^{3} \|(\partial_t, \partial_x)^j u(t,\cdot)\|^2 + \sum_{j=0}^{2} \|(\partial_t, \partial_x)^j (\theta, q)(t,\cdot)\|^2 \leq d_1 \mathrm{e}^{-d_2}\, \Lambda_0.$$

Proof: In a great part, the proof imitates the energy estimates derived for the linearized system in section 3. That is one tries to find a Lyapunov function and appropriate multipliers, respectively, and treats the arising nonlinear terms as perturbations of the energy terms. These perturbations are of cubic type and hence will be regarded as "small" terms. Thus an *a priori* estimate will be proved that simultaneously gives a uniform bound on the highest norms of the local solution such that a continuation argument is possible, as well as allows to conclude the exponential stability.

We start from a local solution according to Theorem 6.1 (with $s = 3$) and derive estimates for $\Lambda(t)$ for t in the time interval of local existence. The multipliers are chosen like in the linearized case, now working with nonlinear multipliers that lead to additional nonlinear terms.

First we define the first-order energy term

$$\mathcal{E}_1(t) := \frac{1}{2}\int_0^1 (kdu_t^2 + kdau_x^2 + kb\theta^2 + bg\tau q^2)(t,x)dx \equiv \mathcal{E}_1(t; u, \theta, q).$$

Then we easily obtain

$$\frac{d}{dt}\mathcal{E}_1 = -\int_0^1 bgq^2 dx + R_c \tag{7.6}$$

with

$$\begin{aligned} R_c \quad = \quad & \frac{1}{2}\int_0^1 (kd)_t u_t^2 - (akd)_x u_x u_t + (akd)_t u_x^2 + (bkd)_x \theta u_t + (kb)_t \theta^2 \\ & + (kbg)_x q\theta + (bg\tau)_t q^2 dx. \end{aligned} \tag{7.7}$$

R_c essentially consists only of cubic terms because of the differentiations to be carried out. Differentiating (7.1)–(7.3) with respect to t once and twice, we obtain in the same way for the second- and third-order energy terms

$$\mathcal{E}_2(t) := \mathcal{E}_1(t; u_t, \theta_t, q_t), \quad \mathcal{E}_3(t) := \mathcal{E}_1(t; u_{tt}, \theta_{tt}, q_{tt}),$$

$$\frac{d}{dt}\mathcal{E}_2 = -\int_0^1 bgq_t^2 dx + R_c \tag{7.8}$$

and

$$\frac{d}{dt}\mathcal{E}_3 = -\int_0^1 bgq_{tt}^2 dx + R_c, \tag{7.9}$$

where we use the same symbol R_c as in (7.6), here and in the sequel, to denote different, but essentially cubic remainder terms. Observe that due to the regularity in the local existence theorem we have to consider also the third-order energy term, while it was sufficient to consider only the first- and second-order energy terms in the linearized case, cp. section 3.

Moreover, observe that $\sum_{j=1}^3 \mathcal{E}_j(t)$ is equivalent to $\Lambda(t)$ defined in Theorem 7.1, that is

$$\exists\, C_1, C_2 > 0\, \forall\, t: \;\; C_1\Lambda(t) \le \sum_{j=1}^3 \mathcal{E}_j(t) \le C_2\Lambda(t). \tag{7.10}$$

The second inequality in (7.10) is obvious, the first can be seen from Poincaré's estimate for u and using the differential equations to estimate successively

$$u_{xx}, \theta_x, q_x, q_{tx}, \theta_{xx}, u_{xxx}, \theta_{xt}, u_{txx}, q_{xx}.$$

Therefore, to prove the theorem, we shall prove an *a priori* estimate for $\sum_{j=1}^3 \mathcal{E}_j(t)$.

From (7.3) and its — with respect to t — differentiated version we conclude

$$\int_0^1 \theta_x^2 + \theta_{xt}^2 dx \le c\int_0^1 q^2 + q_t^2 + q_{tt}^2 dx + R_c, \tag{7.11}$$

where c will denote various positive constants not depending on t or on the initial data (but on K from the assumptions, for instance).

Multiplying the equations (7.1) and (7.2) by $\frac{1}{a}u_{xx}$ and $\frac{3}{ad}u_{tx}$, respectively, we obtain after partial integrations and adding the two resulting equations

$$\int_0^1 \frac{2}{a}u_{tx}^2 dx + \frac{1}{2}\int_0^1 u_{xx}^2 + \frac{d}{dt}\left(\int_0^1 \frac{1}{a}u_{tx}u_x - \frac{3g}{a^2d}qu_{tt} + \frac{3bg\tau}{a^2dk}qq_t + \frac{3bg}{a^2dk}q^2\right.$$
$$\left. + \frac{3\tau}{adk}q_tu_t + \frac{3}{adk}qu_t dx\right) \le -[\frac{3g}{ad}qu_{tx}]_{x=0}^{x=1} + c\int_0^1 q_t^2 + \theta_x^2 dx + R_c. \tag{7.12}$$

Similarly, we obtain after differentiating the differential equations (7.1)–(7.3) with respect to t

$$\int_0^1 \frac{2}{a}u_{txt}^2 dx + \frac{1}{2}\int_0^1 u_{xxt}^2 + \frac{d}{dt}\left(\int_0^1 \frac{1}{a}u_{txt}u_{xt} - \frac{3g}{a^2d}q_tu_{ttt} + \frac{3bg\tau}{a^2dk}q_tq_{tt} + \frac{3bg}{a^2dk}q_t^2\right.$$
$$\left. + \frac{3\tau}{adk}q_{tt}u_{tt} + \frac{3}{adk}q_tu_{tt} dx\right) \le -[\frac{3g}{ad}q_tu_{txt}]_{x=0}^{x=1} + c\int_0^1 q_{tt}^2 + \theta_{xt}^2 dx + R_c. \tag{7.13}$$

Using the Poincaré estimate and the differential equations (7.1)–(7.3) as well as their differentiated (with respect to t) version, we get

$$\int_0^1 u_t^2 + u_{tt}^2 + u_{ttt}^2 + \theta^2 + \theta_t^2 dx \le c\int_0^1 u_{xx}^2 + u_{xxt}^2 + u_{tx}^2 + u_{ttx}^2 + q^2 + q_t^2 + q_{tt}^2 dx + R_c. \tag{7.14}$$

An estimate for u_x is obtained by multiplication of (7.1) by u leading to

$$\int_0^1 u_x^2 dx \leq c \int_0^1 u_{tt}^2 + \theta_x^2 dx + R_c. \tag{7.15}$$

A multiplication of (7.2) by θ_t and of the differentiated equation by θ_{tt}, respectively, yields

$$\int_0^1 \theta_t^2 + \theta_{tt}^2 dx - \frac{d}{dt}\left(\int_0^1 2gq\theta_x + 2gq_t\theta_{xt} dx\right)$$

$$\leq c\int_0^1 q_t^2 + \theta_x^2 + q_{tt}^2 + \theta_{xt}^2 dx + \int_0^1 d^2(u_{tx}^2 + u_{txt}^2)dx + R_c. \tag{7.16}$$

The boundary terms in (7.12) and (7.13) are also treated in a similar way to the linearized case, that is at first we have

$$|[\frac{3g}{ad}(qu_{tx} + q_t u_{txt})]_{x=0}^{x=1}| \leq \frac{c}{\varepsilon_1^3}\int_0^1 q^2 + q_t^2 dx + c_1\varepsilon\int_0^1 \theta_t^2 + \theta_{tt}^2 dx + c_2\int_0^1 u_{tx}^2 + u_{txt}^2 dx$$

$$+\varepsilon_1\left([a(|u_{tx}|^2 + |u_{txt}|^2)]_{x=1} + [a(|u_{tx}|^2 + |u_{txt}|^2)]_{x=0}\right) + R_c, \tag{7.17}$$

where $\varepsilon_1 > 0$ is still arbitrary, and c_1, c_2 denote fixed positive constants, as well as will do $c_3, c_4, \ldots$ in the sequel. Then we differentiate (7.1) with respect to t once and twice and multiply by ϕu_{tx} and ϕu_{txt}, respectively, where

$$\phi(x) := 1 - 2x,$$

and obtain

$$\frac{d}{dt}(\int_0^1 u_{tt}\phi u_{tx} + u_{ttt}\phi u_{txt} dx) + \frac{1}{2}(|u_{tt}(1)|^2 + |u_{tt}(0)|^2 + |u_{ttt}(1)|^2 + |u_{ttt}(0)|^2)$$

$$-\int_0^1 u_{tt}^2 + u_{ttt}^2 dx + \frac{1}{2}[a(|u_{tx}|^2 + |u_{txt}|^2)]_{x=1} + \frac{1}{2}[a(|u_{tx}|^2 + |u_{txt}|^2)]_{x=0}$$

$$-\int_0^1 a(u_{tx}^2 + u_{txt}^2)dx + \int_0^1 \phi b(\theta_{tx}u_{tx} + \theta_{txt}u_{txt})dx = R_c. \tag{7.18}$$

On the other hand we can multiply (7.2) by $-\frac{b}{d}\phi\theta_{tx}$ and the differentiated equation by $-\frac{b}{d}\phi\theta_{txt}$, respectively, and we get

$$\frac{1}{2}[\frac{b}{d}(|\theta_t|^2 + |\theta_{tt}|^2)]_{x=1} + \frac{1}{2}[\frac{b}{d}(|\theta_t|^2 + |\theta_{tt}|^2)]_{x=0} - \int_0^1 \frac{b}{d}(\theta_t^2 + \theta_{tt}^2)dx - \frac{d}{dt}(\int_0^1 \frac{bg}{d}(q_x\phi\theta_x + q_{xt}\phi\theta_{xt})dx$$

$$-\int_0^1 \frac{bg}{d\tau}(q_x\phi\theta_x + q_{xt}\phi\theta_{xt}) + [\frac{gbk}{2d\tau}(|\theta_x|^2 + |\theta_{xt}|^2)]_{x=1} + [\frac{gbk}{2d\tau}(|\theta_x|^2 + |\theta_{xt}|^2)]_{x=0}$$

$$-\int_0^1 \frac{bgk}{d\tau}(\theta_x^2+\theta_{xt}^2)dx - \int_0^1 \phi b(\theta_{tx}u_{tx}+\theta_{txt}u_{txt})dx = R_c. \tag{7.19}$$

Adding (7.18) and (7.19) we obtain

$$\frac{d}{dt}\left(2\varepsilon_1\int_0^1 u_{tt}\phi u_{tx}+u_{ttt}\phi u_{txt}-\frac{bg}{d}(q_x\phi\theta_x+q_{xt}\phi\theta_{xt})dx\right)$$
$$+\varepsilon_1\left([a(|u_{tx}|^2+|u_{txt}|^2)]_{x=1}+[a(|u_{tx}|^2+|u_{txt}|^2)]_{x=0}\right)$$
$$\leq c_3\varepsilon_1\int_0^1 u_{tt}^2+u_{ttt}^2dx+c_4\varepsilon_1\int_0^1 u_{tx}^2+u_{txt}^2dx+c_5\varepsilon_1\int_0^1\theta_t^2+\theta_{tt}^2dx+c\int_0^1\theta_x^2+\theta_{xt}^2dx+R_c. \tag{7.20}$$

Observe that the third-order terms in θ appearing in (7.18) and (7.19) just cancel each other.

A combination of (7.17) and (7.20) yields

$$\begin{aligned}
|[\frac{3g}{ad}(qu_{tx}+q_tu_{txt})]_{x=0}^{x=1}| \quad \leq \quad & \frac{c}{\varepsilon_1^3}\int_0^1 q^2+q_t^2dx+c_6\varepsilon_1\int_0^1\theta_t^2+\theta_{tt}^2dx \\
& +c_7\varepsilon_1\int_0^1 u_{tx}^2+u_{txt}^2dx+c_8\varepsilon_1\int_0^1 u_{tt}^2+u_{ttt}^2dx \\
& +c_9\varepsilon_1\int_0^1\theta_x^2+\theta_{xt}^2dx-\frac{d}{dt}\left(2\varepsilon_1\int_0^1 u_{tt}\phi u_{tx}+u_{ttt}\phi u_{txt}\right. \\
& \left.-\frac{bg}{d}(q_x\phi\theta_x+q_{xt}\phi\theta_{xt})dx\right)+R_c.
\end{aligned} \tag{7.21}$$

Combining (7.12), (7.13), (7.16) and (7.21) we obtain for sufficiently small ε_1

$$c_{10}\int_0^1 u_{tx}^2+u_{txt}^2dx+c_{11}\int_0^1 u_{xx}^2+u_{xxt}^2dx+c_{12}\int_0^1\theta_t^2+\theta_{tt}^2dx+\frac{d}{dt}G(t)\leq c_{13}\int_0^1 q^2+q_t^2+q_{tt}^2dx+R_c, \tag{7.22}$$

where

$$\begin{aligned}
G(t) \quad := \quad & \int_0^1 \frac{1}{a}(u_{tx}u_x+u_{txt}u_{xt})-\frac{3g}{a^2d}(qu_{tt}+q_tu_{ttt})+\frac{3bg\tau}{a^2dk}(qq_t+q_tq_{tt})+\frac{3bg}{a^2dk}(q^2+q_t^2) \\
& +\frac{3\tau}{adk}(q_tu_t+q_{tt}u_{tt})+\frac{3}{adk}(qu_t+q_tu_{tt})+2\varepsilon_1[u_{tt}\phi u_{tx}+u_{ttt}\phi u_{txt} \\
& -\frac{bg}{d}(q_x\phi\theta_x+q_{xt}\phi\theta_{xt})]-c_{14}g(q\theta_x+q_t\theta_{xt})dx.
\end{aligned}$$

As Lyapunov function we may now take, for $\varepsilon>0$,

$$F(t) := \frac{1}{\varepsilon}\sum_{j=1}^{3}\mathcal{E}_j(t)+G(t).$$

For sufficiently small ε we have the following equivalence of $F(t)$ and $\sum_{j=1}^{3} \mathcal{E}_j(t)$,

$$\exists\, C_3, C_4 > 0\, \forall\, t \geq 0: \quad C_3 F(t) \leq \sum_{j=1}^{3} \mathcal{E}_j(t) \leq C_4 F(t), \tag{7.23}$$

and summarizing (7.6), (7.8), (7.9), (7.11), (7.14), (7.15), (7.22), we obtain

$$\frac{d}{dt} F(t) \leq -c_{15} \sum_{j=1}^{3} \mathcal{E}_j(t) + |R_c| \leq -c_{16} F(t) + |R_c|. \tag{7.24}$$

Based on classical calculus for composite functions and Gagliardo-Nirenberg estimates the cubic terms in R_c can be estimated in a manner also done for classical thermoelasticity, see e.g [17, 10], to obtain

$$|R_c| \leq c_{17} (F(t))^{3/2},$$

hence we conclude from (7.24)

$$\frac{d}{dt} F(t) \leq -c_{16} F(t) + c_{17} (F(t))^{3/2}$$

whence we get for sufficiently small $F(0)$, e.g. (cp. page 128 in [10]) if

$$F(0) < \frac{1}{2} \left(\frac{c_{16}}{2c_{17}}\right)^2$$

that

$$F(t) \leq c_{18} \mathrm{e}^{-c_{16} t} F(0). \tag{7.25}$$

Recalling (7.23) and (7.10), the estimate (7.25) together with a continuation argument for the local solution proves the theorem.

Q.E.D.

Remark: We have shown that at least for solutions near the equilibrium state zero, there is no blow-up of solutions in finite time. Smooth solutions exist globally and are exponentially stable. On the other hand, it is known already for the hyperbolic-parabolic system of classical thermoelasticity ($\tau = 0$) that for sufficiently large data smooth solutions will blow up in finite time, see [4, 7, 6]. The more this behavior is expected for the damped, but purely hyperbolic system of thermoelasticity with second sound.

8 Appendix: Linear well-posedness

For the sake of completeness we present a short direct discussion of the well-posedness for the linear initial boundary value problem (2.1)–(2.5). The boundary conditions (3.1) or (4.1) instead of (2.5) can be treated similarly, cp. [13, 10].

We transform the system (2.1)–(2.5) into a first-order system of evolution type, finally applying semigroup theory. Let for a solution (u, q, θ) be V defined as

$$V := \begin{pmatrix} \kappa\delta\alpha u_x \\ u_t \\ \theta \\ q \end{pmatrix}, \qquad V(0, \cdot) \equiv V_0 := \begin{pmatrix} \kappa\delta\alpha u_{0,x} \\ u_1 \\ \theta_0 \\ q_0 \end{pmatrix} \tag{8.1}$$

and let

$$Q := \begin{pmatrix} \frac{1}{\kappa\alpha\delta} & 0 & 0 & 0 \\ 0 & \kappa\delta & 0 & 0 \\ 0 & 0 & \kappa\beta & 0 \\ 0 & 0 & 0 & \gamma\beta\tau_0 \end{pmatrix}, \qquad N := \begin{pmatrix} 0 & -\partial_x & 0 & 0 \\ -\partial_x & 0 & \kappa\delta\beta\partial_x & 0 \\ 0 & \kappa\delta\beta\partial_x & 0 & \gamma\kappa\beta\partial_x \\ 0 & 0 & \gamma\kappa\beta\partial_x & \gamma\beta \end{pmatrix}.$$

Then V satisfies

$$V_t(t) + Q^{-1}NV(t) = 0, \quad V(0) = V_0.$$

Let $\mathcal{H} := \Big(L^2((0,L))\Big)^4$ with inner product

$$\langle V, W\rangle_{\mathcal{H}} := \langle V, QW\rangle_{L^2},$$

let

$$A : D(A) \subset \mathcal{H} \to \mathcal{H}, \quad AV := Q^{-1}NV \text{ for}$$
$$V \in \; D(A) := \left\{V = (V^1, V^2, V^3, V^4)' \in \mathcal{H} \mid V^2, V^4 \in H_0^1((0,L)), V^1, V^3 \in H^1((0,L))\right\},$$

i.e.

$$V_t(t) + AV(t) = 0, \quad V(0) = V_0. \tag{8.2}$$

On the other hand, if V satisfies (8.2) for V_0 defined in (8.1), then

$$u(t,\cdot) := u_0(\cdot) + \int_0^t V^2(s,\cdot)ds, \quad \theta := V^3, \quad q := V^4$$

satisfy (2.1)–(2.5), i.e. (8.2) and (2.1)–(2.5) are equivalent (in the chosen spaces). The well-posedness is now a corollary of the following lemma characterizing A as a generator of a C_0-semigroup of contractions.

Lemma 8.1 *(i) $D(A)$ is dense in $\mathcal{H}$ and $-A$ is dissipative.*

(ii) A is closed.

(iii)

$$D(A^*) = D(A), \quad A^*W = Q^{-1}\begin{pmatrix} 0 & \partial_x & 0 & 0 \\ \partial_x & 0 & -\kappa\delta\beta\partial_x & 0 \\ 0 & -\kappa\delta\beta\partial_x & 0 & -\gamma\kappa\beta\partial_x \\ 0 & 0 & -\gamma\kappa\beta\partial_x & \gamma\beta \end{pmatrix} W.$$

Proof:

(i) The density of $D(A)$ is obvious,

$$\mathrm{Re}\langle -AV, V\rangle_{\mathcal{H}} = -\gamma\beta\|V^4\|_{L^2}^2 \le 0.$$

(ii) Let $(V_n)_n \subset D(A), \quad V_n \to V \in \mathcal{H},\ AV_n \to F \in \mathcal{H}$ as $n \to \infty$.
Then

$$\forall \Phi \in \mathcal{H}: \quad \langle AV_n, \Phi\rangle_{\mathcal{H}} \to \langle F, \Phi\rangle_{\mathcal{H}}.$$

Choosing successively
1. $\Phi = (\Phi^1, 0, 0, 0)',\ \Phi^1 \in H^1((0,L)),$ 2. $\Phi = (0, 0, \Phi^3, 0)',\ \Phi^3 \in H^1((0,L)),$
3. $\Phi = (0, 0, 0, \Phi^4)',\ \Phi^4 \in C_0^\infty((0,L)),$ 4. $\Phi = (0, \Phi^2, 0, 0)',\ \Phi^2 \in C_0^\infty((0,L))$
we obtain
1. $V^2 \in H_0^1((0,L))$ and $-\partial_x V^2 = [QF]^1$ (first component),
2. $V^4 \in H_0^1((0,L))$ and $\kappa\delta\beta\partial_x V^2 + \gamma\kappa\beta\partial_x V^4 = [QF]^3$,
3. $V^3 \in H^1((0,L))$ and $\gamma\kappa\beta\partial_x V^3 + \gamma\beta V^4 = [QF]^4$,
4. $V^1 \in H^1((0,L))$ and $-\partial_x V^1 + \kappa\delta\beta\partial_x V^3 = [QF]^2$,

that is $V \in D(A)$ and $AV = F$.

(iii) $W \in D(A^*) \Leftrightarrow \exists F \in \mathcal{H}\ \forall \Phi \in D(A): \quad \langle A\Phi, W\rangle_{\mathcal{H}} = \langle \Phi, F\rangle_{\mathcal{H}}.$

Choosing Φ appropriately as in the proof if (ii), the conclusion follows.

Q.E.D.

With the Hille-Yosida theorem for C_0-semigroups we thus have

Theorem 8.2 *$-A$ generates a C_0-semigroup of contractions $\left\{e^{-tA}\right\}$ $t \geq 0$. Let $V_0 \in D(A)$. Then the unique solution $V \in C_1([0,\infty), \mathcal{H}) \cap C_0([0,\infty), D(A))$ to (8.2) is given by $V(t) = e^{-tA}V_0$.*
If $V_0 \in D(A^n),\ n \in \mathbb{N}$, then $V \in C_0([0,\infty), D(A^n))$ and (8.2) yields higher regularity in t.

Remark: The spaces and norms have been chosen such that the first order energy E_1 of (u, θ, q) corresponds to the norm of V in $\mathcal{H}$:

$$E_1(t) = \frac{1}{2}\|V(t)\|^2_{\mathcal{H}}.$$

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