Regularity and Stabilization of Magneto-Elastic Systems

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Abstract: We consider the mathematical model for a plate in a bounded reference configuration $\Omega \subset \mathbb{R}^n$, first with $n = 2$, which is interacting with $n = 2$ magnetic fields. The latter have a damping effect. It will be shown that the arising system generates an analytic semigroup and that the estimated exponential decay rate tends to zero if the n constant directing magnetic vectors tend to become linearly dependent. Then, an analogous model for $n = 3$ will be considered. In the case that there are less than n magnetic fields we prove the strong stability exemplarily for cubes.

1 Introduction

We consider the mathematical model for a plate in a bounded reference configuration $\Omega \subset \mathbb{R}^n$, first with $n = 2$, which is interacting with n magnetic fields. The latter have a damping effect which raises the question whether the arising system gives an analytic semigroup or, at least, if through the damping by n magnetic fields, the system becomes exponentially stable. Concerning the latter we will then elaborate that the estimated decay rate tends to zero if the n constant determining magnetic vectors ($\vec{H}^1, \vec{H}^2$, see below) tend to become linearly dependent. Afterwards, an analogous model for $n = 3$ will be considered. If there are less than n magnetic fields, exponential stability or analyticity is not expected. In this case we prove the strong stability exemplarily for the rectangle $\Omega = (-\pi, \pi)^2$ in two dimensions, resp. for the cube $\Omega = (-\pi, \pi)^3$ in three dimensions.

The model we study in two dimensions, $n = 2$, is given by

$$\rho_0 u_{tt} + d\Delta^2 u - \alpha_1(\text{rot rot}\, h^1) \cdot \vec{H}^1 - \alpha_2(\text{rot rot}\, h^2) \cdot \vec{H}^2 = 0 \quad \text{in} \quad \Omega \times [0, \infty), \tag{1.1}$$

$$\rho_1 h_t^1 + \text{rot rot}\, h^1 + \beta_1 \text{rot rot}\, (u_t \vec{H}^1) = 0 \quad \text{in} \quad \Omega \times [0, \infty), \tag{1.2}$$

$$\rho_2 h_t^2 + \text{rot rot}\, h^2 + \beta_2 \text{rot rot}\, (u_t \vec{H}^2) = 0 \quad \text{in} \quad \Omega \times [0, \infty), \tag{1.3}$$

$$(j = 1, 2 :) \quad \text{div}\, h^j = 0 \quad \text{in} \quad \Omega \times [0, \infty), \tag{1.4}$$

with boundary conditions

$$u = \Delta u = 0, \quad \nu \times h^j = 0, \quad j = 1, 2, \tag{1.5}$$

and initial conditions

$$u(\cdot, 0) = u_0, \quad u_t(\cdot, 0) = u_1, \quad h^j(\cdot, 0) = h_0^j, \quad j = 1, 2. \tag{1.6}$$

[0]AMS subject classification: 35 M 13, 5 35 B 40, 74 F 15

Keywords and phrases: magneto-elasticity, exponential stability, analyticity

Here, $\Omega \subset \mathbb{R}^2$ is a smoothly bounded, simply connected domain, and $u \in \mathbb{R}$ and $h^j \in \mathbb{R}^2$, $j = 1, 2$, denote the displacement resp. the magnetic fields. Here we assume that $\rho_0\, \rho_1, \rho_2, d, \alpha_1, \alpha_2, \beta_1, \beta_2$ are positive constants. The vectors $\vec{H}^1,\ \vec{H}^2 \in \mathbb{R}^2$ are constant and are assumed to satisfy:

$$\vec{H}^1 \text{ and } \vec{H}^2 \text{ are linearly independent.} \tag{1.7}$$

In other words, we assume

$$D_2 := \det\left(\vec{H}^1, \vec{H}^2\right) \neq 0. \tag{1.8}$$

The meaning of " rot " is here as follows:

$$F : \mathbb{R}^2 \longrightarrow \mathbb{R}^2, \qquad \text{then} \qquad \operatorname{rot} F := \partial_1 F_2 - \partial_2 F_1,$$

$$f : \mathbb{R}^2 \longrightarrow \mathbb{R}, \qquad \text{then} \qquad \operatorname{rot} f := (\partial_2 f, -\partial_1 f)'.$$

The vector product is correspondingly given by

$$\nu \times h^j = \nu_1\, h_2^j - \nu_2\, h_1^j.$$

Then it holds, as in three dimensions,

$$\Delta = \nabla \operatorname{div} - \operatorname{rot} \operatorname{rot}.$$

We remark that condition (1.4) could be derived from the differential equations assuming it only for the initial value, i.e. at $t = 0$. From the mathematical point of view system (1.1)–(1.4) is a plate equation coupled with a parabolic equation for the magnetic field.

The case of *only one* magnetic field, i.e. $h^2 \equiv 0$, for which one can find the modeling e.g. in [5, 8, 17], has found attention before, also with nonlinear models, but not yet much. The polynomial stability has been proved in [12] for the boundary conditions $\nu \cdot h^j = 0$, $\nu \times \operatorname{rot} h^j = 0$, $u = \partial u / \partial \nu = 0$. With our contribution we initiate to investigate the analyticity and exponential stability of the system, which are open questions for this fourth-second-order coupled system.

This model is closely related to the thermoelastic plate model, where we have the temperature instead of the magnetic field, and where the parabolic equation is, up to the coupling term, the classical heat equation. The main difference, and also the main problem, consists in the coupling terms. The thermoelastic coupling term makes it possible to show that the corresponding thermoelastic semigroup is analytic, see [10]. Instead, it is not known whether the semigroup associated to the magneto-elastic plate here, with one magnetic field, is analytic or at least exponentially stable in bounded domains. Here it will be our task to investigate the situation and the impact of two magnetic fields and answer these questions.

The model with one magnetic field is also related to the *second*-order magneto-elastic model, where the fourth-order bi-Laplacean Δ^2 is replaced by the second-order operator elasticity $-\mu\Delta - (\lambda + \mu)\nabla \operatorname{div}$. Here the strong stability – the energy decays to zero for each fixed initial data, but not necessarily uniformly – was proved in [19], while the polynomial stability was proved for a class of two- and three-dimensional domains in [14, 16]. On the other hand, the lack of exponential stability was shown in [4], using microlocal analysis. We remark that also the Cauchy problem, i.e. for $\Omega = \mathbb{R}^n$, has been studied, and polynomial decay rates were obtained, see [1, 13].

In $n = 3$ space dimensions, the system of equations turns into

$$\rho_0 u_{tt} + d\Delta^2 u - \alpha_1 \operatorname{rot}\operatorname{rot} h^1 \cdot \vec{H}^1 - \alpha_2 \operatorname{rot}\operatorname{rot} h^2 \cdot \vec{H}^2 - \alpha_3 \operatorname{rot}\operatorname{rot} h^3 \cdot \vec{H}^3 = 0, \tag{1.9}$$

$$\rho_1 h^1_t + \operatorname{rot}\operatorname{rot} h^1 + \beta_1 \operatorname{rot}\operatorname{rot}(u_t \vec{H}^1) = 0, \tag{1.10}$$

$$\rho_2 h^2_t + \operatorname{rot}\operatorname{rot} h^2 + \beta_2 \operatorname{rot}\operatorname{rot}(u_t \vec{H}^2) = 0, \tag{1.11}$$

$$\rho_3 h^3_t + \operatorname{rot}\operatorname{rot} h^3 + \beta_3 \operatorname{rot}\operatorname{rot}(u_t \vec{H}^3) = 0, \tag{1.12}$$

$$(j = 1,2,3:) \quad \operatorname{div} h^j = 0, \tag{1.13}$$

with boundary conditions

$$u = \Delta u = 0, \quad \nu \times h^j = 0, \quad j = 1,2,3, \tag{1.14}$$

and initial conditions

$$u(\cdot,0) = u_0, \quad u_t(\cdot,0) = u_1, \quad h^j(\cdot,0) = h^j_0, \quad j = 1,2,3. \tag{1.15}$$

Here, $\Omega \subset \mathbb{R}^3$ is a smoothly bounded, simply connected domain. Our main results are, for both dimensions $n = 2,3$:

- n magnetic fields are dealt with.
- The analyticity of the semigroup and, consequently, the exponential stability are obtained: Theorem 3.7 and Corollary 3.8.
- As a consequence of the analyticity and the compactness of the resolvent operators compactness of the semigroup is proved: Corollary 3.9.
- The exponential stability via multiplier methods providing information on the dependence of the rate of the decay in terms of the degree of linear independence of the vectors $\vec{H}^j$, j=1,2(,3) is obtained: Theorem 4.2 and Theorem 5.1.
- The strong stability in the case of less than n magnetic fields is proved for cubes: Theorem 6.2.

We remark that, given the exponential stability property and and the compactness of the semigroup of the semigroup at hand, nonlinear problems for small or even large data, depending on the nonlinearity, seem accessible.

The paper is organized as follows. In Section 2 we provide the well-posedness. In Section 3, the analyticity of the associated semigroup is proved using semigroup arguments and estimates on the resolvents along the imaginary axes, also giving the exponential stability. With further sophisticated estimates, the exponential stability is then proved in Section 4 ($n = 2$) and in

Section 5 ($n = 3$) once more, now using multiplier (energy) methods, allowing a characterization of the decay rate in terms of the determinants D_2 and D_3, respectively. In Section 6 the strong stability is proved in the situation of less than n magnetic fields for a square resp. a cube.

Throughout the paper, we use standard notation, in particular we use the Sobolev spaces $L^2 = L^2(\Omega)$, and $H^s = W^{s,2}(\Omega)$, $s \in \mathbb{N}_0$, with their associated norms $\|\cdot\|$ respectively $\|\cdot\|_{H^s}$. For the inner product in L^2 we use the notation $\langle\cdot,\cdot\rangle$, and we write ∂_j short for $\frac{\partial}{\partial x_j}$, and ∂_t for $\frac{\partial}{\partial t}$.

2 Well-posedness

In order to formulate the problem as evolution equation of first-order in time, we consider in $n = 2$ space dimensions as phase space

$$\mathcal{H} := \big(H^2(\Omega) \cap H^1_0(\Omega)\big) \times L^2(\Omega) \times L^2_*(\Omega) \times L^2_*(\Omega),$$

where

$$L^2_* := \{\, h \in (L^2)^n, \mid \operatorname{div} h = 0\} \quad (n = 2,3).$$

Taking the norm

$$\|V\|^2_{\mathcal{H}} := \int_\Omega \big[\frac{d}{\rho_0}|\Delta u|^2 + |v|^2 + \frac{\alpha_1\rho_1}{\beta_1\rho_0}|h^1|^2 + \frac{\alpha_2\rho_2}{\beta_2\rho_0}|h^2|^2\big]\,dx \tag{2.1}$$

for

$$V := \begin{pmatrix} u \\ v \\ h^1 \\ h^2 \end{pmatrix}.$$

and the associated inner product $(\cdot,\cdot)_{\mathcal{H}}$, $\mathcal{H}$ is a Hilbert space.

In $n = 3$ space dimensions we have analogously

$$\mathcal{H} := \big(H^2(\Omega) \cap H^1_0(\Omega)\big) \times L^2(\Omega) \times L^2_*(\Omega) \times L^2_*(\Omega) \times L^2_*(\Omega)$$

and

$$\|V\|^2_{\mathcal{H}} := \int_\Omega \big[\frac{d}{\rho_0}|\Delta u|^2 + |v|^2 + \frac{\alpha_1\rho_1}{\beta_1\rho_0}|h^1|^2 + \frac{\alpha_2\rho_2}{\beta_2\rho_0}|h^2|^2 + \frac{\alpha_3\rho_3}{\beta_3\rho_0}|h^3|^2\big]\,dx. \tag{2.2}$$

For two space dimensions we introduce the operator $\mathcal{A}$ by

$$\mathcal{A}V := \begin{pmatrix} v \\ -\frac{d}{\rho_0}\Delta^2 u + \frac{\alpha_1}{\rho_0}(\operatorname{rot}\operatorname{rot} h^1)\cdot\vec{H}^1 + \frac{\alpha_2}{\rho_0}(\operatorname{rot}\operatorname{rot} h^2)\cdot\vec{H}^2 \\ -\frac{\beta_1}{\rho_1}\operatorname{rot}\operatorname{rot}(v\vec{H}^1) - \frac{1}{\rho_1}\operatorname{rot}\operatorname{rot} h^1 \\ -\frac{\beta_2}{\rho_2}\operatorname{rot}\operatorname{rot}(v\vec{H}^2) - \frac{1}{\rho_2}\operatorname{rot}\operatorname{rot} h^2 \end{pmatrix},$$

and in three space dimensions by

$$\mathcal{A}V := \begin{pmatrix} v \\ -\frac{d}{\rho_0}\Delta^2 u + \frac{\alpha_1}{\rho_0}(\operatorname{rot}\operatorname{rot} h^1)\cdot \vec{H}^1 + \frac{\alpha_2}{\rho_0}(\operatorname{rot}\operatorname{rot} h^2)\cdot \vec{H}^2 + \frac{\alpha_3}{\rho_0}(\operatorname{rot}\operatorname{rot} h^3)\cdot \vec{H}^2 \\ -\frac{\beta_1}{\rho_1}\operatorname{rot}\operatorname{rot}(v\vec{H}^1) - \frac{1}{\rho_1}\operatorname{rot}\operatorname{rot} h^1 \\ -\frac{\beta_2}{\rho_2}\operatorname{rot}\operatorname{rot}(v\vec{H}^2) - \frac{1}{\rho_2}\operatorname{rot}\operatorname{rot} h^2 \\ -\frac{\beta_3}{\rho_3}\operatorname{rot}\operatorname{rot}(v\vec{H}^3) - \frac{1}{\rho_3}\operatorname{rot}\operatorname{rot} h^3 \end{pmatrix}$$

for V in the domain

$$\mathcal{D}(\mathcal{A}) := \{\, V \in \mathcal{H} \,|\, u \in H^4, \Delta u \in H_0^1, h^j \in H^2, \nu \times h^j = 0 \text{ on } \partial\Omega,\ j = 1,2(,3)\,\}. \tag{2.3}$$

The original initial-boundary value problem now turns into the first-order evolution equation

$$V_t(t) = \mathcal{A}V(t), \quad V(0) = V_0, \tag{2.4}$$

where

$$V(t) := \begin{pmatrix} u(\cdot,t) \\ u_t(\cdot,t) \\ h^1(\cdot,t) \\ h^2(\cdot,t) \\ [h^3(\cdot,t)] \end{pmatrix}, \qquad V_0 := \begin{pmatrix} u_0 \\ u_1 \\ h_0^1 \\ h_0^2 \\ [h_0^3] \end{pmatrix}.$$

The operator $\mathcal{A}$ is densely defined and dissipative, it satisfies

$$\operatorname{Re}(\mathcal{A}V,V)_{\mathcal{H}} = -\frac{\alpha_1}{\beta_1\rho_0}\int_\Omega |\operatorname{rot} h^1|^2 dx - \frac{\alpha_2}{\beta_2\rho_0}\int_\Omega |\operatorname{rot} h|^2 \left[-\frac{\alpha_3}{\beta_3\rho_0}\int_\Omega |\operatorname{rot} h|^3 dx\right] \le 0. \tag{2.5}$$

The properties of the model that we deal with here depend on the properties of the resolvent operator over the imaginary axes, that is defined by the system

$$i\lambda U - \mathcal{A}U = F$$

for $\lambda \in \mathbb{R}$. In terms of its components the above system is given by

$$i\lambda u - v = f_1, \tag{2.6}$$

$$i\lambda u + \frac{d}{\rho_0}\Delta^2 u - \frac{\alpha_1}{\rho_0}(\operatorname{rot}\operatorname{rot} h^1)\cdot\vec{H}^1 - \frac{\alpha_2}{\rho_0}(\operatorname{rot}\operatorname{rot} h^2)\cdot\vec{H}^2 = f_2, \tag{2.7}$$

$$i\lambda h^1 + \frac{\beta_1}{\rho_1}\operatorname{rot}\operatorname{rot}(v\vec{H}^1) + \frac{1}{\rho_1}\operatorname{rot}\operatorname{rot} h^1 = f_3, \tag{2.8}$$

$$i\lambda h^2 + \frac{\beta_2}{\rho_2}\operatorname{rot}\operatorname{rot}(v\vec{H}^2) + \frac{1}{\rho_2}\operatorname{rot}\operatorname{rot} h^2 = f_4. \tag{2.9}$$

Now we show that $\lambda = 0 \in \rho(\mathcal{A})$ (resolvent set). For this purpose let $F \in \mathcal{H}$, and we look for a solution $V \in D(\mathcal{A})$ to $\mathcal{A}V = F = (f_1, f_2, f_3, f_4)$, and suppose w.l.o.g. $n = 2$. So let $v := -f_1$, then

$$v \in H^2 \cap H_0^1, \qquad \|v\|_{H^2} \le c\|F\|_{\mathcal{H}}, \tag{2.10}$$

with $c > 0$ not depending on F. Now we look for $j = 1, 2$, at the Maxwell system

$$\begin{aligned} -\frac{1}{\rho_j}\,\text{rot rot}\, h^j &= -f_{2+j} + \frac{\beta_j}{\rho_j}\,\text{rot rot}\,(v\vec{H}^j) \\ \text{div}\, h^j &= 0, \\ \nu \times h^j &= 0 \qquad (\partial\Omega). \end{aligned}$$

This is uniquely solvable in a simply connected domain with smooth boundary and using (2.10),

$$h^j \in H^2, \qquad \|h^j\|_{H^2} \leq c\|F\|_{\mathcal{H}}.$$

For the last estimate we used the inequality

$$\|h^j\|_{H^2} \leq c\|\,\text{rot rot}\, h^j\|, \tag{2.11}$$

cf. [20]. Finally, we look at

$$-\frac{d}{\rho_0}\Delta^2 u = -f_2 - \frac{\alpha_1}{\rho_0}(\,\text{rot rot}\, h^1)\cdot\vec{H}^1 - \frac{\alpha_2}{\rho_0}(\,\text{rot rot}\, h^2)\cdot\vec{H}^2,$$

$$u = \Delta u = 0 \qquad (\partial\Omega).$$

This now has a unique solution satisfying

$$u \in H^4 \cap H_0^1, \quad \Delta u \in H_0^1, \qquad \|u\|_{H^4} \leq c\|F\|_{\mathcal{H}}.$$

Altogether we have found a unique $V \in D(\mathcal{A})$ solving $\mathcal{A}V = F$ and satisfying $\|V\|_{\mathcal{H}} \leq c\|F\|_{\mathcal{H}}$.

Remark 2.1. *The last estimates imply, using Rellich's compactness theorem, that the inverse $\mathcal{A}^{-1}$ is compact.*

Remark 2.2. *By (2.5) the solution V to $i\lambda V - \mathcal{A}V = F$, for any $\lambda \in \mathbb{R}$, satisfies*

$$\frac{\alpha_1}{\beta_1\rho_0}\int_\Omega |\,\text{rot}\, h^1|^2 dx + \frac{\alpha_2}{\beta_2\rho_0}\int_\Omega |\,\text{rot}\, h|^2 \left[+\frac{\alpha_3}{\beta_3\rho_0}\int_\Omega |\,\text{rot}\, h|^3 dx\right] = Re\,(V,F)_{\mathcal{H}} \tag{2.12}$$

By the Lumer-Phillips theorem we conclude

Theorem 2.3. *The operator $\mathcal{A}$ is the infinitesimal generator of a C_0-semigroup of contractions $\left(\mathrm{e}^{t\mathcal{A}}\right)_{t\geq 0}$ on $\mathcal{H}$. For any initial data $V_0 \in D(\mathcal{A})$, problem (2.4) has a unique solution V satisfying*

$$V \in C^1([0,\infty),\mathcal{H}) \cap C^0([0,\infty), D(\mathcal{A})).$$

3 Analyticity of the semigroup

We will prove the analyticity for two and three space dimensions using the following characterization that can be found in [18, 11].

Theorem 3.1. *A contraction semigroup $T=(T(t))_{t\geq 0}$ generated by $\mathbb{A}$ on a Hilbert space $\mathcal{H}$ is analytic if and only if the following two conditions holds.*

$$i\mathbb{R}\subset\varrho(\mathbb{A}) \tag{3.1}$$

$$\lim_{\lambda\in\mathbb{R},\ |\lambda|\to\infty}\sup|\lambda|\|(i\lambda-\mathcal{A})^{-1}\|<\infty \tag{3.2}$$

Lemma 3.2. *Under the above notations the operator $\mathcal{A}$ associated to system* (1.1)-(1.4) *(respectively* (1.9)-(1.13)*) satisfies condition* (3.1)*.*

PROOF: Since $\mathcal{A}^{-1}$ is compact it is enough to show that there are no imaginary eigenvalues. Suppose that there exist $U\in D(\mathcal{A})$ satisfying

$$i\lambda U-\mathcal{A}U=0$$

we conclude by (2.12) that $h^j=0$, for $j=1,2(,3)$. Using (2.8) and (2.9) we get

$$\text{rot rot}\,(v\vec{H}^j)=0\quad j=1,2(,3).$$

Now, in two dimensions, this is equivalent to the linear system

$$\underbrace{\begin{pmatrix} 0 & -H_2^1 & 0 & H_1^1\\ H_2^1 & 0 & -H_1^1 & 0\\ 0 & -H_2^2 & 0 & H_1^2\\ H_2^2 & 0 & -H_2^1 & 0\end{pmatrix}}_{:=\mathbb{H}_2}\begin{pmatrix}\partial_1^2 v\\ \partial_2\partial_1 v\\ \partial_1\partial_2 v\\ \partial_2^2 v\end{pmatrix}=0, \tag{3.3}$$

Since $\det\mathbb{H}_2=D_2^2\neq 0$ we conclude $v=0$. Using (2.6) we get that $u=0$ so we have that $U=0$, proving (3.1) in the two-dimensional case.

In three dimensions, we obtain analogously

$$\underbrace{\begin{pmatrix} 0 & -H_2^1 & -H_3^1 & 0 & H_1^1 & 0 & 0 & 0 & H_1^1\\ H_2^1 & 0 & 0 & -H_1^1 & 0 & -H_3^1 & 0 & 0 & H_2^1\\ H_3^1 & 0 & 0 & 0 & H_3^1 & 0 & -H_1^1 & -H_2^1 & 0\\ 0 & -H_1^2 & -H_3^2 & 0 & H_1^2 & 0 & 0 & 0 & H_1^2\\ H_2^2 & 0 & 0 & -H_1^2 & 0 & -H_3^2 & 0 & 0 & H_2^2\\ H_3^2 & 0 & 0 & 0 & H_3^2 & 0 & -H_1^2 & -H_2^2 & 0\\ 0 & -H_2^3 & -H_3^3 & 0 & H_1^3 & 0 & 0 & 0 & H_1^3\\ H_2^3 & 0 & 0 & -H_1^3 & 0 & -H_3^3 & 0 & 0 & H_2^3\\ H_3^3 & 0 & 0 & 0 & H_3^3 & 0 & -H_1^3 & -H_2^3 & 0\end{pmatrix}}_{:=\mathbb{H}_3}\begin{pmatrix}\partial_1^2 v\\ \partial_1\partial_2 v\\ \partial_1\partial_3 v\\ \partial_2\partial_1 v\\ \partial_2^2 v\\ \partial_2\partial_3 v\\ \partial_3\partial_1 v\\ \partial_3\partial_2 v\\ \partial_3^2 v\end{pmatrix}=0, \tag{3.4}$$

satisfies (checked by Maple©)

$$\det\mathbb{H}_3=-2(D_3)^3\neq 0.$$

So we get again $v = 0$, then $u = 0$ and finally $U = 0$, thus proving (3.1) also in the three-dimensional case.
We remark that, with (3.1), we now already have the *strong stability* of the semigroup.
To prove (3.2) we start with

Lemma 3.3. *For λ large enough the following inequalities hold*

$$\begin{aligned} \|\Delta^2 u\|_{L^2} &\le c|\lambda| \left(\|v\|_{L^2} + \|h^i\|_{L^2} + \|\Delta u\|_{L^2} + \frac{1}{|\lambda|}\|F\|_{\mathcal{H}} \right) \equiv c|\lambda|\mathfrak{E}, && (3.5) \\ \|\operatorname{rot}\operatorname{rot} h^j\|^2 &\le c|\lambda| \left(\|h^i\|_{L^2} + \|\Delta u\|_{L^2} + \frac{1}{|\lambda|}\|F\|_{\mathcal{H}} \right), \qquad j = 1, 2(,3). && (3.6) \end{aligned}$$

Moreover we have

$$\|u\|_{H^1} \le \frac{c}{|\lambda|^{1/2}}\|U\|_{\mathcal{H}} + \frac{c}{|\lambda|^{1/2}}\|U\|_{\mathcal{H}}^{1/2}\|F\|_{\mathcal{H}}^{1/2}, \quad \|v\|_{H^1} \le c|\lambda|^{1/2}\|U\|_{\mathcal{H}} + c\|U\|_{\mathcal{H}}^{1/2}\|F\|_{\mathcal{H}}^{1/2}, \quad (3.7)$$

where c is a positive constant not depending on λ.

PROOF: We consider the two-dimensional case, the three-dimensional case is similar. From (2.7) we get

$$\|\Delta^2 u\|_{L^2} \le c\|i\lambda v\|_{L^2} + c\|(\operatorname{rot}\operatorname{rot} h^1)\cdot \vec{H}^1\|_{L^2} + c\|(\operatorname{rot}\operatorname{rot} h^2)\cdot \vec{H}^2\|_{L^2} + c\|F\|_{\mathcal{H}}. \quad (3.8)$$

By (2.8) and (2.9) we have

$$\|(\operatorname{rot}\operatorname{rot} h^j)\cdot \vec{H}^1\|_{L^2} \le c|\lambda| \left(\|h^j\|_{L^2} + \|\Delta u\|_{L^2}\right) + c\|F\|_{\mathcal{H}}, \qquad j = 1, 2(,3).$$

Inserting this inequality into (3.8) we get

$$\|\Delta^2 u\|_{L^2} \le c|\lambda| \left(\|v\|_{L^2} + \|h^1\|_{L^2} + \|h^2\|_{L^2} + \|\Delta u\|_{L^2}\right) + c\|F\|_{\mathcal{H}}. \quad (3.9)$$

Using the same ideas we get (3.6), hence the first part of this Lemma follows. Using interpolation once more we get

$$\begin{aligned} \|u\|_{H^1} &\le c\|u\|_{L^2}^{1/2}\|u\|_{H^2}^{1/2} \\ &\le \frac{c}{|\lambda|^{1/2}}\|v - f_1\|_{L^2}^{1/2}\|u\|_{H^2}^{1/2} \\ &\le \frac{c}{|\lambda|^{1/2}}\|v\|_{L^2}^{1/2}\|u\|_{H^2}^{1/2} + \frac{c}{|\lambda|^{1/2}}\|f_1\|_{L^2}^{1/2}\|u\|_{H^2}^{1/2}. \end{aligned}$$

Similarly,

$$\begin{aligned} \|v\|_{H^1} &\le c\|v\|_{L^2}^{1/2}\|v\|_{H^2}^{1/2} \\ &\le c\|v\|_{L^2}^{1/2}\|i\lambda\Delta u - \Delta f_1\|_{L^2}^{1/2} \\ &\le c|\lambda|^{1/2}\|U\|_{\mathcal{H}} + c\|U\|_{\mathcal{H}}^{1/2}\|F\|_{\mathcal{H}}^{1/2}. \end{aligned}$$

Hence our conclusion follows. □

Lemma 3.4. *For any $\epsilon > 0$ there is $c_\epsilon > 0$ such that for $j = 1, 2$*

$$\|h^j\| \leq \epsilon \|U\|_{\mathcal{H}} + \frac{c_\epsilon}{|\lambda|} \|F\|_{\mathcal{H}}, \tag{3.10}$$

where $c_\epsilon \to \infty$ as $\epsilon \to 0$.

PROOF: For any f_j such that $\operatorname{div} f_j = 0$ there exists exactly one solution to

$$i\lambda h^j_f + \operatorname{rot}\operatorname{rot} h^j_f = f_j \quad \text{in} \quad \Omega, \qquad \nu \times h^j_f = 0 \quad \text{on} \quad \partial\Omega, \tag{3.11}$$

cf. [9]. Let us decompose the function h^j into two components:

$$h^j = h^j_f + (h^j - h^j_f) \equiv h^j_f + h^j_r.$$

Since $\nu \times h^j = \nu \times h^j_f = 0$ we get $\nu \times h^j_r = 0$. Moreover, taking the difference of equation (2.8) with (3.11) with (and (2.9) also) we get

$$i\lambda h^j_r + \operatorname{rot}\operatorname{rot} h^j_r + \beta_j \operatorname{rot}\operatorname{rot}(v\vec{H}^j) \quad = \quad 0. \tag{3.12}$$

Multiplying (3.11) by $\overline{i\lambda h^j_f}$ and taking the real part, then multiplying (3.11) by $\overline{\operatorname{rot}\operatorname{rot} h^j_f}$ and taking the real part we get

$$\|\lambda h^j_f\|_{L^2} \leq c\|F\|_{\mathcal{H}}, \quad \|\operatorname{rot}\operatorname{rot} h^j_f\|_{L^2} \leq \|F\|_{\mathcal{H}}. \tag{3.13}$$

Multiplying (3.11) by $\overline{i\lambda h^j_f}$ and taking the imaginary part and using the first inequality in (3.13) we get

$$|\lambda|^{1/2} \|\operatorname{rot} h^j_f\|_{L^2} \leq c\|F\|. \tag{3.14}$$

Using equation (3.12) we conclude

$$|\lambda| \|h^j_r\|_{H^{-1}} \quad \leq \quad c\|\operatorname{rot} h^j_r\| + c\|v\|_{H^1}.$$

Recalling that $h^j = h^j_f + h^j_r$ and using (3.14) we get

$$\|\operatorname{rot} h^i_r\| = \|\operatorname{rot} h^r - \operatorname{rot} h^r_f\| \leq \|\operatorname{rot} h^r\| + |\lambda|^{-1/2}\|F\|.$$

From Lemma 3.3 we get $\|v\|_{H^1} \leq c|\lambda|^{1/2}\|U\|_{\mathcal{H}} + c\|U\|^{1/2}_{\mathcal{H}}\|F\|^{1/2}_{\mathcal{H}}$, and using (2.12) we arrive at

$$|\lambda| \|h^j_r\|_{H^{-1}} \quad \leq \quad c|\lambda|^{1/2}\|U\|_{\mathcal{H}} + c\|U\|^{1/2}_{\mathcal{H}}\|F\|^{1/2}_{\mathcal{H}} + |\lambda|^{-1/2}\|F\|. \tag{3.15}$$

Using interpolation and recalling that $\|v\| \leq \|U\|_{\mathcal{H}}$ we get

$$\|h^j_r\|_{L^2} \quad \leq \quad c\|h^j_r\|^{1/2}_{H^{-1}} \|\operatorname{rot} h^j_r\|^{1/2}_{L^2}.$$

Using (3.15), (2.12) and (3.14) we get

$$\begin{aligned}
\|h^j_r\|_{L^2} \quad &\leq \quad \frac{c}{|\lambda|^{1/2}} \left(|\lambda|^{1/2}\|U\|_{\mathcal{H}} + c\|U\|^{1/2}_{\mathcal{H}}\|F\|^{1/2}_{\mathcal{H}} + |\lambda|^{-1/2}\|F\|\right)^{1/2} (\|\operatorname{rot} h\|_{L^2} + |\lambda|^{-1/2}\|F\|)^{1/2} \\
&\leq \quad \frac{c}{|\lambda|^{1/2}} \left(|\lambda|^{1/2}\|U\|_{\mathcal{H}} + c\|U\|^{1/2}_{\mathcal{H}}\|F\|^{1/2}_{\mathcal{H}} + |\lambda|^{-1/2}\|F\|\right)^{1/2} (\|U\|^{1/2}_{\mathcal{H}}\|F\|^{1/2}_{\mathcal{H}} + |\lambda|^{-1/2}\|F\|)^{1/2} \\
&\leq \quad \epsilon\|U\|_{\mathcal{H}} + \frac{\epsilon}{|\lambda|}\|F\|_{\mathcal{H}}.
\end{aligned}$$

Hence, using (3.13), we obtain

$$\|h^j\| \leq \|h_r^j\|_{L^2} + \|h_f^j\|_{L^2} \leq \epsilon \|U\|_{\mathcal{H}} + \frac{c_\epsilon}{|\lambda|}\|F\|_{\mathcal{H}}$$

and our conclusion follows. $\square$

Next we have the estimate

Lemma 3.5. *For the solution of the resolvent equation, we have*

$$\|\Delta u\|^2 \quad \leq \quad \epsilon \|U\|_{\mathcal{H}}^2 + \frac{c_\epsilon}{|\lambda|^2}\|F\|_{\mathcal{H}}^2. \tag{3.16}$$

PROOF: From (2.8) and (2.9) we have

$$\begin{aligned} i\lambda h^1 + \text{ rot rot}\, h^1 + \beta_1 i\lambda \,\text{rot rot}\,(u\vec{H}^1) &= f_3 + \beta_1 \,\text{rot rot}\,(f_1\vec{H}^1), \\ i\lambda h^2 + \text{ rot rot}\, h^2 + \beta_2 i\lambda \,\text{rot rot}\,(u\vec{H}^2) &= f_4 + \beta_2 \,\text{rot rot}\,(f_1\vec{H}^2), \end{aligned}$$

leading to

$$\beta_1 \,\text{rot rot}\,(u\vec{H}^1) \quad = \quad -h^1 - \frac{1}{i\lambda}\text{rot rot}\, h^1 + \frac{1}{i\lambda}F_1, \tag{3.17}$$

$$\beta_2 \,\text{rot rot}\,(u\vec{H}^2) \quad = \quad -h^2 - \frac{1}{i\lambda}\text{rot rot}\, h^2 + \frac{1}{i\lambda}F_2, \tag{3.18}$$

where

$$F_1 := \Big(f_3 + \beta_1 \,\text{rot rot}\,(f_1\vec{H}^1)\Big), \quad F_2 := \Big(f_4 + \beta_2 \,\text{rot rot}\,(f_1\vec{H}^2)\,).$$

Multiplying these equations by $\overline{\text{rot rot}\,(u\vec{H}^j)}$ for $j = 1, 2$, we get

$$\begin{aligned} \beta_1 \int_\Omega |\,\text{rot rot}\,(u\vec{H}^j)|^2\, dx &= -\int_\Omega \text{rot rot}\,(u\vec{H}^j) h^j\, dx - \frac{1}{i\lambda}\int_\Omega \text{rot rot}\,(u\vec{H}^j)\,\text{rot rot}\, h^j\, dx \\ &\quad + \frac{1}{i\lambda}\int_\Omega \text{rot rot}\,(u\vec{H}^j) F_j\, dx \\ &= -\int_\Omega \text{rot rot}\,(u\vec{H}^j) h^j\, dx - \frac{1}{i\lambda}\int_\Omega \text{rot rot rot}\,(u\vec{H}^j)\,\text{rot}\, h^j\, dx \\ &\quad \underbrace{- \frac{1}{i\lambda}\int_{\partial\Omega} \text{rot rot}\,(u\vec{H}^j)\nu \times \text{ rot}\, h^j\, d\Gamma}_{=:J} + \underbrace{\frac{1}{i\lambda}\int_\Omega \text{rot rot}\,(u\vec{H}^j) F_j\, dx}_{\leq \frac{c}{|\lambda|}\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}} \end{aligned}$$

Hence

$$\begin{aligned} \int_\Omega |\,\text{rot rot}\,(u\vec{H}^j)|^2\, dx \quad &\leq \quad c\int_\Omega |h^j|^2\, dx + \frac{\epsilon}{|\lambda|}\int_\Omega |\,\text{rot rot rot}\,(u\vec{H}^j)|^2\, dx + \\ &\qquad J + \frac{c_\epsilon}{|\lambda|}\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}. \end{aligned} \tag{3.19}$$

Using interpolation it follows

$$\begin{aligned} \int_\Omega |\,\text{rot rot rot}\,(u\vec{H}^j)|^2\, dx \quad &\leq \quad c\|u\|_{H^2}\|u\|_{H^4} \\ &\leq \quad c\|\Delta u\|_{L^2}\|\Delta^2 u\|_{L^2}. \end{aligned}$$

Using Lemma 3.4 we get

$$\int_\Omega |\operatorname{rot}\operatorname{rot}\operatorname{rot}(u\vec{H}^j)|^2\,dx \quad\leq\quad c|\lambda|\|\Delta u\|_{L^2}\left(\|v\|_{L^2}+\|h^j\|_{L^2}+\|\Delta u\|_{L^2}\right)+c\|\Delta u\|_{L^2}\|F\|_{\mathcal{H}}.$$

Note that

$$\begin{aligned} J &\leq \frac{c}{|\lambda|}\|u\|_{H^{5/2}}\|h^j\|_{H^{3/2}} \\ &\leq \frac{c}{|\lambda|}\left(\|u\|_{H^2}^{3/4}\|u\|_{H^4}^{1/4}\right)\left(\|h^i\|_{H^1}^{1/2}\|h^j\|_{H^2}^{1/2}\right) \\ &\leq \frac{c}{|\lambda|}\left(\|\Delta u\|_{L^2}^{3/4}\|\Delta^2 u\|_{L^2}^{1/4}\right)\left(\|h^j\|_{H^1}^{1/2}\|h^i\|_{H^2}^{1/2}\right). \end{aligned}$$

Recalling the definition of $\mathfrak{E}$ given in (3.5) and using Lemma 3.3 we obtain

$$\begin{aligned} J &\leq \frac{c}{|\lambda|^{1/4}}\left(\|\Delta u\|_{L^2}^{3/4}\mathfrak{E}^{1/4}\right)\left(\|h^i\|_{H^1}^{1/2}\mathfrak{E}^{1/2}\right) \\ &\leq \frac{c}{|\lambda|^{1/4}}\mathfrak{E}^{3/2}\|h^i\|_{H^1}^{1/2} \\ &\leq \epsilon\mathfrak{E}^2+\frac{c_\epsilon}{|\lambda|}\|h^i\|_{H^1}^2. \end{aligned}$$

Inserting this inequality into (3.19) we get

$$\begin{aligned} \int_\Omega |\operatorname{rot}\operatorname{rot}(u\vec{H}^j)|^2\,dx \quad\leq\quad & c\int_\Omega |h^j|^2\,dx+\epsilon\int_\Omega |v|^2+|\Delta u|^2\,dx+\frac{c_\epsilon}{|\lambda|}\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}} \\ & +\frac{c_\epsilon}{|\lambda|^2}\|F\|_{\mathcal{H}}^2. \end{aligned} \tag{3.20}$$

Since $\vec{H}^1$ and $\vec{H}^2$ are linearly independent, we conclude from (3.20) for $j=1,2$

$$\int_\Omega |\Delta u|^2\,dx \leq c\int_\Omega |h^1|^2+|h^2|^2\,dx+\epsilon\int_\Omega |v|^2\,dx+\frac{c_\epsilon}{|\lambda|}\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}+\frac{c_\epsilon}{|\lambda|^2}\|F\|_{\mathcal{H}}^2.$$

Now using Lemma 3.4 our conclusion follows. □

Finally we estimate v starting with

Lemma 3.6. *Under the above conditions we have*

$$\int_\Omega |v|^2\,dx \leq \epsilon\|U\|_{\mathcal{H}}^2+\frac{c_\epsilon}{|\lambda|^2}\|F\|_{\mathcal{H}}^2.$$

PROOF: Multiplying equation (2.7) by $\overline{i\lambda v}$ we get

$$\int_\Omega |\lambda v|^2\,dx+d\int_\Omega \Delta u\overline{i\lambda\Delta v}\,dx+\alpha_1\int_\Omega \operatorname{rot}h^1\cdot\operatorname{rot}(\overline{i\lambda v}H^1)\,dx+$$

$$\alpha_2\int_\Omega(\operatorname{rot}h^2)\cdot\operatorname{rot}(\overline{i\lambda v}H^2)\,dx=\int_\Omega f_2\overline{i\lambda v}\,dx.$$

Using (2.6) we arrive at

$$\begin{aligned}
\int_\Omega |\lambda v|^2\,dx &= di\lambda\int_\Omega \Delta u\overline{\Delta v}\,dx \underbrace{+i\lambda\alpha_1\int_\Omega h^1\cdot \text{rot rot}\,(\overline{v}H^1)\,dx+\alpha_2 i\lambda\int_\Omega (h^2)\cdot\text{rot rot}\,(\overline{v}H^2)\,dx}_{\le c|\lambda|^2\|\mathbf{h}\|^2+c|\lambda|^2\|\Delta u\|^2+c|\lambda|\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}}\\
&\quad +\int_\Omega f_2\overline{i\lambda v}\,dx\\
&\le c|\lambda|^2\int_\Omega |\Delta u^2|\,dx+c|\lambda|^2\int_\Omega |h^1|^2+|h^2|^2\,dx+c|\lambda|\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}, \qquad (3.21)
\end{aligned}$$

where we used

$$\begin{aligned}
\left|i\lambda\alpha_j\int_\Omega h^j\cdot\text{rot rot}\,(\overline{v}H^j)\,dx\right| &= \left|i\lambda\alpha_j\int_\Omega h^j\cdot\text{rot rot}\,(\overline{(i\lambda u-f_1)}H^j)\,dx\right|\\
&\le c|\lambda|^2\|h^j\|_{L^2}\|\,\text{rot rot}\,u\|_{L^2}+c|\lambda|\|h^j\|_{L^2}\|\,\text{rot rot}\,f_1\|_{L^2}\\
&\le c|\lambda|^2\|h^i\|_{L^2}\|\Delta u\|_{L^2}+c|\lambda|\|h^i\|_{L^2}\|\Delta f_1\|_{L^2}\\
&\le c|\lambda|^2\|\mathbf{h}\|^2+c|\lambda|^2\|\Delta u\|^2+c|\lambda|\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}.
\end{aligned}$$

From Lemma 3.4, Lemma 3.5 and (3.21) we get

$$\int_\Omega |\lambda v|^2\,dx\le \epsilon|\lambda|^2\|U\|^2_{\mathcal{H}}+c_\epsilon\|F\|^2_{\mathcal{H}},$$

finishing the proof. □

Now we can prove the analyticity.

Theorem 3.7. *The semigroup associated to system* (1.1)-(1.4) *is analytic.*

PROOF: From Lemma 3.4, Lemma 3.5 and Lemma 3.6 we get

$$\int_\Omega |v|^2+|\Delta u|^2+|h^1|^2+|h^2|^2\,dx\le\epsilon\|U\|^2_{\mathcal{H}}+\frac{\epsilon}{|\lambda|^2}\|F\|^2_{\mathcal{H}}.$$

This is equivalent to

$$\|U\|^2_{\mathcal{H}}\le\epsilon\|U\|^2_{\mathcal{H}}+\frac{c_\epsilon}{|\lambda|^2}\|F\|^2_{\mathcal{H}}.$$

So we have

$$\|U\|^2_{\mathcal{H}}\le\frac{c_\epsilon}{|\lambda|^2}\|F\|^2_{\mathcal{H}}$$

which implies

$$\|(i\lambda I-\mathcal{A})^{-1}F\|^2_{\mathcal{H}}\le\frac{c_\epsilon}{|\lambda|^2}\|F\|^2_{\mathcal{H}}.$$

Using Lemma 3.2 and Theorem 3.1 our conclusion follows. □

As usual we can conclude

Corollary 3.8. *The semigroup associated to system* (1.1)-(1.4) *is exponentially stable, i.e. there are $k>0$ and $C>0$ sucht that for all $V_0\in\mathcal{H}$ and all $t\ge 0$ we have*

$$\|\mathrm{e}^{t\mathcal{A}}V_0\|_{\mathcal{H}}\le C\|V_0\|_{\mathcal{H}}\,\mathrm{e}^{-kt}. \qquad (3.22)$$

Corollary 3.9. *The semigroup* $(e^{\mathcal{A}t})_{t\geq 0}$ *associated to system* (1.1)-(1.4) *is uniformly continuous for* $t > 0$ *and a compact operator over the phase space* $\mathcal{H}$.

PROOF: Since the semigroup is analytic, in particular it is a differentiable semigroup, that is the semigroup is differentiable infinitely many times in the uniform operator topology for $t > 0$. This implies that the semigroup is uniformly continuous. Finally, using Remark 2.1 and Theorem 3.3 of [18, p. 48] we get that the semigroup is compact. □

In the next two sections, we will prove the exponential stability by the energy (multiplier) method directly, in particular in order to get information on a possible dependence of the decay rate k in Corollary 3.8 on the degree of linear independence of the vectors $\vec{H}^j$, $j = 1, 2(, 3)$, respectively in terms of the determinants D_2 and D_3.

4 The exponential decay rate and the magnetic vectors $\vec{H}^1, \vec{H}^2$

We consider the system in two space dimensions from the introduction, (1.1)-(1.6), i.e.

$$\begin{aligned} \rho_0 u_{tt} + d\Delta^2 u - \alpha_1 \,\mathrm{rot\ rot}\, h^1 \cdot \vec{H}^1 - \alpha_2 \,\mathrm{rot\ rot}\, h^2 \cdot \vec{H}^2 &= 0, && (4.1)\\ \rho_1 h^1_t + \mathrm{rot\ rot}\, h^1 + \beta_1 \,\mathrm{rot\ rot}\,(u_t \vec{H}^1) &= 0, && (4.2)\\ \rho_2 h^2_t + \mathrm{rot\ rot}\, h^2 + \beta_2 \,\mathrm{rot\ rot}\,(u_t \vec{H}^2) &= 0, && (4.3)\\ (j = 1, 2:) \quad \mathrm{div}\, h^j &= 0, && (4.4) \end{aligned}$$

with boundary conditions

$$u = \Delta u = 0, \quad \nu \times h^j = 0, \quad j = 1, 2, \tag{4.5}$$

and initial conditions

$$u(\cdot, 0) = u_0, \quad u_t(\cdot, 0) = u^1, \quad h^j(\cdot, 0) = h^j_0, \quad j = 1, 2. \tag{4.6}$$

To show the exponential stability, let us introduce the first order energy

$$E_1(t) := \frac{d}{\rho_0}\|\Delta u(t,\cdot)\|^2 + \|u_t(t,\cdot)\|^2 + \frac{\alpha_1\rho_1}{\beta_1\rho_0}\|h^1(t,\cdot)\|^2 + \frac{\alpha_2\rho_2}{\beta_2\rho_0}\|h^2(t,\cdot)\|^2$$

Then by denoting

$$E_1(t) \equiv E_1(u, h^1, h^2; t)$$

For initial data taking in $D(\mathcal{A})$ we define the second-order term E_2 by

$$E_2(t) := E_1(u_t, h^1_t, h^2_t; t) = \frac{d}{\rho_0}\|\Delta u_t(t,\cdot)\|^2 + \|u_{tt}(t,\cdot)\|^2 + \frac{\alpha_1\rho_1}{\beta_1\rho_0}\|h^1_t(t,\cdot)\|^2 + \frac{\alpha_2\rho_2}{\beta_2\rho_0}\|h^2_t(t,\cdot)\|^2,$$

arising as first-order term for the differentiated (in time) system (4.1)–(4.3). For the sum

$$E_{nd}(t) := E_1(t) + E_2(t) \tag{4.7}$$

the exponential decay will be proved. Let $D_2 = \det\begin{pmatrix} \vec{H}^1 \, \vec{H}^2 \end{pmatrix}$ again.

Using multiplicative techniques, we easily get, cf. (2.5),

$$\begin{aligned}\frac{d}{dt}E_{nd}(t) &= -\frac{2\alpha_1}{\rho_0\beta_1}\left(\|\operatorname{rot} h^1\|^2 + \|\operatorname{rot} h_t^1\|^2\right) - \frac{2\alpha_2}{\rho_0\beta_2}\left(\|\operatorname{rot} h^2\|^2 + \|\operatorname{rot} h_t^2\|^2\right) \\ &\leq -c\left(\|h^1\|_{H^1}^2 + \|h^2\|_{H^1}^2 + \|h_t^1\|_{H^1}^2 + \|h_t^2\|_{H^1}^2\right), \qquad (4.8)\end{aligned}$$

where we used the simple connectedness for the last inequality, see [3, p. 356] or [9, p.157], and where we mostly drop the parameters t, x. Moreover, c will denote generic positive constants not depending on D_2 (as $D_2 \to 0$).

Let us introduce the functional

$$\mathfrak{J} := -\rho_1\left(\langle h^1, u_t\,\vec{H}^1\rangle + \langle h^2, u_t\,\vec{H}^2\rangle\right).$$

Lemma 4.1. *Under the above notations we have for any $\varepsilon > 0$*

$$\begin{aligned}\frac{d}{dt}\mathfrak{J} &\leq -c|D_2|^2\left(\|u_t\|^2 + \|\nabla u_t\|^2\right) + \varepsilon\frac{\rho_1}{\rho_0}\|\nabla\Delta u\|^2 + \varepsilon\left(\|u_{tt}\|^2 + \|\Delta u_t\|^2\right) + \\ &\quad + \frac{c}{\varepsilon}\left(\|h^1\|_{H^1}^2 + \|h^2\|_{H^1}^2\right) + c\sum_{k=1}^{2}\|h_t^k\|^2. \qquad (4.9)\end{aligned}$$

Proof: Multiplication of (4.2) by $u_t\,\vec{H}^1$ resp. of (4.3) by $u_t\,\vec{H}^2$ yields, for $j = 1, 2$,

$$-\rho_1\frac{d}{dt}\langle h^j, u_t\,\vec{H}^j\rangle = \langle \operatorname{rot} h^j, \operatorname{rot}(u_t\,\vec{H}^j)\rangle - \beta_j\|\operatorname{rot}(u_t\,\vec{H}^j)\|^2 - \rho_1\langle h^j, u_{tt}\,\vec{H}^j\rangle. \qquad (4.10)$$

Using the differential equation (4.1) for u_{tt}, we have

$$\begin{aligned}\rho_1\langle h^j, u_{tt}\,\vec{H}^j\rangle &= -\rho_1\langle h^j, \frac{d}{\rho_0}\Delta^2 u\,\vec{H}^j\rangle + \frac{\rho_1\alpha_1}{\rho_0}\langle h^j, \operatorname{rot}\operatorname{rot} h^1\cdot\vec{H}^1\,\vec{H}^j\rangle + \\ &\quad \frac{\rho_1\alpha_2}{\rho_0}\langle h^j, \operatorname{rot}\operatorname{rot} h^2\cdot\vec{H}^2\,\vec{H}^j\rangle \\ &= \underbrace{-\frac{d\rho_1}{\rho_0}\int_{\partial\Omega} h^j\cdot\vec{H}^j\,\overline{\frac{\partial}{\partial\nu}\Delta u}\,ds}_{=:I_R^{0,j}} + \frac{d\rho_1}{\rho_0}\langle\nabla(h^j\cdot\vec{H}^j), \nabla\Delta u\rangle + \\ &\quad \sum_{k=1}^{2}\frac{\rho_k\alpha_k}{\rho_0}\langle\left(\operatorname{rot}(h_1^j\,\vec{H}_1^j) + \operatorname{rot}(h_2^j\,\vec{H}_2^j)\right)\cdot\vec{H}^k, \operatorname{rot} h^k\rangle + \\ &\quad \sum_{k=1}^{2}\underbrace{\frac{\rho_k\alpha_k}{\rho_0}\int_{\partial\Omega}\left(\nu_2\vec{H}_1^k - \nu_1\vec{H}^k\right)(\operatorname{rot} h^k)\,\overline{(h^j\cdot\vec{H}^j)}\,ds}_{=:I_R^{k,j}}. \qquad (4.11)\end{aligned}$$

Thus,

$$-\rho_1\langle h^j, u_{tt}\,\vec{H}^j\rangle \leq \varepsilon\frac{\rho_1}{\rho_0}\|\nabla\Delta u\|^2 + \frac{c}{\varepsilon}\|h^j\|_{H^1}^2 + \sum_{k=0,j=1}^{2}|I_R^{k,j}|. \qquad (4.12)$$

Combining (4.10) and (4.12), we obtain

$$
\begin{aligned}
-\frac{d}{dt}\rho_1\left(\langle h^1, u_t\,\vec{H}^1\rangle + \langle h^2, u_t\,\vec{H}^2\rangle\right) \;&\le\; -\frac{\beta_1}{2}\left(\|\operatorname{rot}(u_t\,\vec{H}^1)\|^2 + \|\operatorname{rot}(u_t\,\vec{H}^2)\|^2\right) + \\
&\quad \frac{c}{\varepsilon}\left(\|h^1\|^2_{H^1} + \|h^2\|^2_{H^1}\right) \\
&\quad + \varepsilon\frac{\rho_1}{\rho_0}\|\nabla\Delta u\|^2 + \sum_{k=0,j=1}^{2}|I_R^{k,j}|. \qquad (4.13)
\end{aligned}
$$

The boundary terms are estimated for $j = 1, 2$ as follows.

$$
|I_R^{0,j}| \;=\; |\frac{d\rho_1}{\rho_0}\int_{\partial\Omega} h^j \cdot \vec{H}^j\,\overline{\frac{\partial}{\partial\nu}\Delta u}\;ds| \le c\|h^j\|_{H^1}\|\nabla\Delta u\|_{H^1} \le c\|h^j\|_{H^1}\|\Delta^2 u\|.
$$

Using the differential equations (4.1)-(4.3) we obtain

$$
\begin{aligned}
|I_R^{0,j}| \;&\le\; c\|h^j\|_{H^1}\left(\|u_{tt}\| + \sum_{k=1}^{2}\|\operatorname{rot}\operatorname{rot} h^k\|\right) \\
&\le\; c\|h^j\|_{H^1}\left(\|u_{tt}\| + \sum_{k=1}^{2}\left(\|h_t^k\| + \|\operatorname{rot}\operatorname{rot}(u_t\vec{H}^k)\|\right)\right) \\
&\le\; c\|h^j\|_{H^1}\sum_{k=1}^{2}\|h_t^k\| + c\|h^j\|_{H^1}\left(\|u_{tt}\| + \|\Delta u_t\|\right).
\end{aligned}
$$

This implies

$$
|I_R^{0,j}| \le c\sum_{k=1}^{2}\|h_t^k\|^2 + \frac{c}{\varepsilon}\|h^j\|^2_{H^1} + \varepsilon\|u_{tt}\|^2 + \varepsilon\|\Delta u_t\|^2. \qquad (4.14)
$$

Moreover,

$$
\begin{aligned}
|I_R^{1,j}| + |I_R^{2,j}| \;&\le\; \sum_{k=1}^{2}\left|\frac{\rho_k\alpha_k}{\rho_0}\int_{\partial\Omega}\left(\nu_2\vec{H}_1^k - \nu_1\vec{H}^k\right)(\operatorname{rot} h^k)\,\overline{(h^j\cdot\vec{H}^j)}\;ds\right| \\
&\le\; c\left(\sum_{k=1}^{2}\|\operatorname{rot} h^k\|_{H^1}\right)\|h^j\|_{H^1} \\
&\le\; c\sum_{k=1}^{2}\|h_t^k\|^2 + \frac{c}{\varepsilon}\|h^j\|^2_{H^1} + \varepsilon\|\Delta u_t\|^2. \qquad (4.15)
\end{aligned}
$$

In the last estimate we used (2.11) again. The negative terms

$$
-\frac{\beta_1}{2}\left(\|\operatorname{rot}(u_t\,\vec{H}^1)\|^2 + \|\operatorname{rot}(u_t\,\vec{H}^2)\|^2\right)
$$

in (4.13) yield a negative term $-\|\nabla u_t\|^2$ as follows. Let, for $k = 1, 2$,

$$
f_k \;:=\; \operatorname{rot}(u_t\vec{H}^k) = \partial_1 u_t\vec{H}_2^k - \partial_2 u_t\vec{H}_1^k.
$$

Considering the above identity as system for the unknowns $\partial_1 u_t$ and $\partial_2 u_t$, we conclude, for $m = 1, 2$, since $D_2 = \det\left(\vec{H}^1\;\vec{H}^2\right) \neq 0$,

$$
\partial_m u_t = \frac{1}{D_2}\left(\vec{H}_m^1 f_2 - \vec{H}_m^2 f_1\right), \qquad (4.16)
$$

hence

$$|\nabla u_t|^2 \leq \frac{c}{|D_2|^2}\left(|\operatorname{rot}(u_t\vec{H}^1)|^2 + |\operatorname{rot}(u_t\vec{H}^2)|^2,\right)$$

implying

$$c\left(\|u_t\|^2 + \|\nabla u_t\|^2\right) \leq \frac{1}{|D_2|^2}\left(\|\operatorname{rot}(u_t\vec{H}^1)\|^2 + \|\operatorname{rot}(u_t\vec{H}^2)\|^2\right). \tag{4.17}$$

Combining (4.13), (4.14), (4.15) and (4.17) and recalling the definition of $\mathfrak{J}$ we obtain inequality (4.9). Hence our conclusion follows. □

Theorem 4.2. *There exist $K_2 > 0$ and $\kappa_2 > 0$ such that for the solution to (4.1) – (4.6) and all $t \geq 0$*

$$E_{nd}(t) \leq K_2 E_{nd}(0)\mathrm{e}^{-\kappa_2 t}$$

holds. $\kappa_2 = \kappa_2(D_2) = \mathcal{O}\left(|D_2|^2\right)$ as $D_2 \to 0$.

As we can see from the proportionality, the decay rate κ_2 vanishes if $\vec{H}^1$ and $\vec{H}^2$ become parallel, and it is strongest if $\vec{H}^1$ and $\vec{H}^2$ are orthogonal to each other. Actually, $|D_2|$ describes the volume of the parallelogram spanned by $\vec{H}^1$ and $\vec{H}^2$ and equals for these unit vectors $|\sin(\varphi)|$, where φ is the angle between the vectors.

Proof of Theorem 4.2: The multiplier used are a modification of those also used for a thermoelastic system in [15], while essential new problems appear here arising through the magnetic field.

Now we multiply (4.1) by Δu to get

$$\begin{aligned}\frac{d}{dt}((\rho_0\langle u_t, \Delta u\rangle) &= -\rho_0\|u_t\|^2 + d\|\nabla\Delta u\|^2 + \sum_{k=1}^{2}\alpha_k\langle\left(\operatorname{rot}\operatorname{rot} h^k\right)\vec{H}^k, \Delta u\rangle \\ &= -\rho_0\|u_t\|^2 + d\|\nabla\Delta u\|^2 + \sum_{k=1}^{2}\alpha_k\langle\operatorname{rot} h^k, \operatorname{rot}(\Delta u\vec{H}^k)\rangle\end{aligned}$$

whence we obtain

$$\frac{d}{dt}((-\rho_0\langle u_t, \Delta u\rangle) \quad \leq \quad \rho_0\|u_t\|^2 - \frac{d}{2}\|\nabla\Delta u\|^2 + c\left(\|h^1\|^2_{H^1} + \|h^2\|^2_{H^1}\right). \tag{4.18}$$

Using Lemma 4.1 we have that

$$\begin{aligned}\frac{d}{dt}\left(\mathfrak{J}(t) - \frac{c}{2}|D_2|^2\langle u_t, \Delta u\rangle\right) \quad \leq \quad & -\frac{c}{2}|D_2|^2\left(\|u_t\|^2 + \|\nabla u_t\|^2 + \frac{d}{2\rho_0}\|\nabla\Delta u\|^2\right) + \\ & +\varepsilon\left(\|u_{tt}\|^2 + \|\Delta u_t\|^2\right) + \frac{c}{\varepsilon}\sum_{k=1}^{2}\|h^k\|^2_{H^1} \\ & +c\sum_{k=1}^{2}\|h^k_t\|^2\end{aligned} \tag{4.19}$$

Now we multiply (4.1) by u_{tt} and obtain

$$\begin{aligned} 0 &= \rho_0\|u_{tt}\|^2 + d\langle \Delta^2 u, u_{tt}\rangle - \langle \alpha_1 \,\mathrm{rot}\ \mathrm{rot}\, h^1 \cdot \vec{H}^1 + \alpha_2 \,\mathrm{rot}\ \mathrm{rot}\, h^2 \cdot \vec{H}^2, u_{tt}\rangle \\ &= \rho_0\|u_{tt}\|^2 + \frac{d}{dt}\langle d\Delta u, \Delta u_t\rangle - d\|\Delta u_t\|^2 - \\ &\quad \frac{d}{dt}\Big(\langle \alpha_1 \,\mathrm{rot}\, h^1,\ \mathrm{rot}\,(u_t\vec{H}^1)\rangle + \langle \alpha_2 \,\mathrm{rot}\, h^2,\ \mathrm{rot}\,(u_t\vec{H}^2)\rangle\Big) + \\ &\quad \langle \alpha_1 \,\mathrm{rot}\, h_t^1,\ \mathrm{rot}\,(u_t\vec{H}^1)\rangle + \langle \alpha_2 \,\mathrm{rot}\, h_t^2,\ \mathrm{rot}\,(u_t\vec{H}^2)\rangle. \end{aligned}$$

Hence

$$\begin{aligned} &\frac{d}{dt}\left(\langle d\Delta u, \Delta u_t\rangle + \sum_{k=1}^{2}\langle \alpha_k \,\mathrm{rot}\, h^k,\ \mathrm{rot}\,(u_t\vec{H}^k)\rangle\right) \\ &\qquad \le -\rho_0\|u_{tt}\|^2 + d\|\Delta u_t\|^2 + \sum_{k=1}^{2}\Big(\varepsilon\|\,\mathrm{rot}\,(u_t\vec{H}^k\|^2 + \frac{c}{\varepsilon}\|h_t^k\|^2\Big). \end{aligned} \tag{4.20}$$

Let us consider the functional

$$\mathfrak{J}_1 := \mathfrak{J}(t) - \frac{c}{2}|D_2|^2\langle u_t, \Delta u\rangle + |D_2|\langle d\Delta u, \Delta u_t\rangle + |D_2|\sum_{k=1}^{2}\langle \alpha_k \,\mathrm{rot}\, h^k,\ \mathrm{rot}\,(u_t\vec{H}^k)\rangle.$$

From (4.19) and (4.20) we get

$$\begin{aligned} \frac{d}{dt}\mathfrak{J}_1(t) \ &\le\ -\frac{c}{4}|D_2|^2\left(\|u_t\|^2 + \|\nabla u_t\|^2 + \frac{d}{2\rho_0}\|\nabla\Delta u\|^2\right) - \frac{\rho_0}{2}|D_2|\|u_{tt}\|^2 + \\ &\quad + \frac{3d}{2}|D_2|\|\Delta u_t\|^2 + \frac{c}{\varepsilon}\sum_{k=1}^{2}\|h^k\|_{H^1}^2 + c\sum_{k=1}^{2}\|h_t^k\|^2, \end{aligned} \tag{4.21}$$

for $\varepsilon < d/2$ small enough. To finally obtain the yet missing term of type "$-\|\Delta u_t\|^2$", we calculate Δu_t in terms of h_t^j and $\mathrm{rot}\ \mathrm{rot}\, h^j$, $j = 1, 2$, as follows. The equations (4.2), (4.3) yield, for $j = 1, 2$,

$$\mathrm{rot}\ \mathrm{rot}\,(u_t\vec{H}^j) = -\frac{\rho_j}{\beta_j}h_t^j - \frac{1}{\beta_j}\,\mathrm{rot}\ \mathrm{rot}\, h^j =: b^j, \tag{4.22}$$

in particular

$$b^j = \begin{pmatrix} \partial_2\partial_1 u_t H_2^j - \partial_2^2 u_t H_1^j \\ -\partial_1^2 u_t H_2^j + \partial_1\partial_2 u_t H_1^j \end{pmatrix}. \tag{4.23}$$

With the matrix $\mathbb{H}_2$, defined in (3.3) and satisfying $\det \mathbb{H}_2 = D_2^2$, (4.22) is equivalent to the linear system

$$\mathbb{H}_2\begin{pmatrix} \partial_1^2 u_t \\ \partial_2\partial_1 u_t \\ \partial_1\partial_2 u_t \\ \partial_2^2 u_t \end{pmatrix} = \begin{pmatrix} b_1^1 \\ b_2^1 \\ b_1^2 \\ b_2^2 \end{pmatrix}. \tag{4.24}$$

By Cramer's rule we compute

$$\partial_1^2 u_t = \frac{1}{D_2}\left(H_1^1 b_2^2 - H_1^2 b_2^1\right), \qquad \partial_2^2 u_t = \frac{1}{D_2}\left(H_2^2 b_1^1 - H_2^1 b_1^2\right),$$

thus

$$D_2 \Delta u_t = H_1^1 f_2^2 - H_1^2 f_2^1 + H_2^2 f_1^1 - H_2^1 f_1^2 =: g. \tag{4.25}$$

Typical terms in g have the form

$$\text{(a)}\ \ a_1 \partial_t h_k^j \qquad \text{and} \qquad \text{(b)}\ \ a_2 \partial_l \partial_m h_k^j, \tag{4.26}$$

where $j, k, l, m = 1, 2$, and the constants a_1, a_2 can be bounded independent of $\vec{H}^j$. Without loss of generality we may assume $D_2 > 0$ (otherwise multiply (4.25) by -1), then we obtain from multiplying (4.25) by $-\Delta u_t$

$$-|D_2| \|\Delta u_t\|^2 = -\langle g, \Delta u_t \rangle. \tag{4.27}$$

The term of type (a) in g is estimated by

$$|a_1 \langle \partial_t h_k^j, \Delta u_t \rangle| \leq \varepsilon_8 \|\Delta u_t\|^2 + c \|\partial_t h_k^j\|^2. \tag{4.28}$$

For type (b) we get

$$\begin{aligned} a_2 \langle \partial_l \partial_m h_k^j, \Delta u_t \rangle &= a_2 \langle \partial_m h_k^j, \partial_l \Delta u_t \rangle \\ &= \frac{d}{dt}\Big(a_2 \langle \partial_m h_k^j, \partial_l \Delta u \rangle \Big) + a_2 \langle \partial_m \partial_t h_k^j, \partial_l \Delta u \rangle. \end{aligned} \tag{4.29}$$

Combining (4.27), (4.28), (4.29) we obtain

$$-|D_2| \|\Delta u_t\|^2 = -\sum a_1 \langle \partial_t h_k^j, \Delta u_t \rangle - \frac{d}{dt} \underbrace{\sum \Big(a_2 \langle \partial_m h_k^j, \partial_l \Delta u \rangle \Big)}_{:= J} + \sum a_2 \langle \partial_m \partial_t h_k^j, \partial_l \Delta u \rangle.$$

For the sum $J = J(t)$ of all - say: P - terms of type $-a_2 \langle \partial_m h_k^j, \partial_l \Delta u \rangle$ we have

$$\frac{d}{dt} J \leq -\frac{|D_2|}{2} \|\Delta u_t\|^2 + \varepsilon \|\nabla \Delta u\|^2 + c_\epsilon \left(\|h_t^1\|_{H^1}^2 + \|h_t^2\|_{H^1}^2 \right). \tag{4.30}$$

Using the differential equations, we obtain

$$\begin{aligned} |J| &\leq c \left(\|h^1\|_{H^1} + \|h^2\|_{H^1} \right) \|\nabla \Delta u\| \\ &\leq c \left(\|h^1\|_{H^1} + \|h^2\|_{H^1} \right) \left(\|\Delta u\| + \|\Delta^2 u\| \right) \\ &\leq c E_{nd} \end{aligned} \tag{4.31}$$

Now we can define the Lyapunov functional

$$\mathcal{L}_2 = \mathcal{L}_2(t) := M E_{nd} + \mathfrak{J}_1 + 4dJ, \tag{4.32}$$

where $M > 0$ will be chosen large enough below. Combining the estimates (4.8), (4.9), (4.18), (4.20), we get

$$\frac{d}{dt}\mathcal{L}_2(t) \leq -\frac{c}{4}|D_2|^2\left(\|u_t\|^2+\|\nabla u_t\|^2+\frac{d}{2\rho_0}\|\nabla\Delta u\|^2\right)-\frac{\rho_0}{2}|D_2|\|u_{tt}\|^2+ -\frac{d}{2}|D_2|\|\Delta u_t\|^2-(M-\frac{c}{\varepsilon})\sum_{k=1}^{2}\|h^k\|_{H^1}^2+(M-c)\sum_{k=1}^{2}\|h_t^k\|^2. \tag{4.33}$$

With these choices and using the Poincaré estimate $\|\Delta u\|\leq c\,\|\nabla\Delta u\|$, we conclude

$$\begin{aligned}\frac{d}{dt}\mathcal{L}_2 &\leq -\min\left\{\frac{M}{2},\frac{c|D_2|^2}{8},\frac{c\rho_0|D_2|^2}{4},\frac{1}{2},\frac{|D_2|\rho_0}{2}\right\}E_{nd}\\ &\equiv -rE_{nd}.\end{aligned} \tag{4.34}$$

Here

$$r=\mathcal{O}\left(|D_2|^2\right) \qquad \text{as } D_2\to 0. \tag{4.35}$$

Moreover, there exists $M_2>0$ such that $Q:=\mathcal{L}_2-ME_{nd}$ satisfies

$$|Q|\leq M_2E_{nd}.$$

Choosing $M\geq 2M_2$ we get

$$M_2E_{nd}\leq\mathcal{L}_2\leq 3M_2E_{nd}. \tag{4.36}$$

Defining

$$\kappa_2:=\frac{r}{3M_2} \qquad \left(=\mathcal{O}\left(|D_2|^2\right)\right)$$

we obtain from (4.34), (4.36)

$$\frac{d}{dt}\mathcal{L}_2\leq-\kappa_2\mathcal{L}_2,$$

thus

$$\mathcal{L}_2(t)\leq\mathcal{L}_2(0)\mathrm{e}^{-\kappa_2\,t},$$

and, again by (4.36),

$$E_{nd}(t)\leq 3M_2E_{nd}(0)\mathrm{e}^{-\kappa_2\,t},$$

which proves Theorem 4.2. □

We remark that one also obtains the exponential decay of the first-order energy term $E_1(t)$ from Theorem 4.2 by an abstract semigroup argument.

Corollary 4.3. *There exists $\tilde{K}_2>0$ such that for the solution to (4.1) –(4.6) and all $t\geq 0$*

$$E_1(t)\leq\tilde{K}_2E_1(0)\mathrm{e}^{-\kappa_2\,t}$$

holds.

Proof: We have

$$E_1=2\|V\|_{\mathcal{H}}^2 \qquad \text{for } V=(u,u_t,h^1,h^2).$$

Let $V^0 \in D(A)$ and $V_t = AV$, $V(0) = V^0$. Then $W^0 := A^{-1}V^0 \in D(A^2)$. Let W satisfy $W_t = AW$, $W(0) = W^0$. Then

$$\|V(t)\|^2_{\mathcal{H}} = \|AW(t)\|^2_{\mathcal{H}} = \|W_t(t)\|^2_{\mathcal{H}} \leq c\left(\|W^0\|^2_{\mathcal{H}} + \|AW^0\|^2_{\mathcal{H}}\right) \mathrm{e}^{-\kappa_2 t}$$

by Theorem 4.2. Thus,

$$\|V(t)\|^2_{\mathcal{H}} \leq c\|V^0\|^2_{\mathcal{H}}\mathrm{e}^{-\kappa_2 t}.$$

□

5 The exponential decay rate in 3-dimensions and the magnetic vectors $\vec{H}^1, \vec{H}^2, \vec{H}^3$

In three space dimensions we have system (1.9)-(1.15) from the introduction,

$$\rho_0 u_{tt} + d\Delta^2 u - \alpha_1 \operatorname{rot}\operatorname{rot} h^1 \cdot \vec{H}^1 - \alpha_2 \operatorname{rot}\operatorname{rot} h^2 \cdot \vec{H}^2 \alpha_3 \operatorname{rot}\operatorname{rot} h^3 \cdot \vec{H}^3 = 0, \tag{5.1}$$

$$\rho_1 h^1_t + \operatorname{rot}\operatorname{rot} h^1 + \beta_1 \operatorname{rot}\operatorname{rot}(u_t \vec{H}^1) = 0, \tag{5.2}$$

$$\rho_2 h^2_t + \operatorname{rot}\operatorname{rot} h^2 + \beta_2 \operatorname{rot}\operatorname{rot}(u_t \vec{H}^2) = 0, \tag{5.3}$$

$$\rho_3 h^3_t + \operatorname{rot}\operatorname{rot} h^3 + \beta_3 \operatorname{rot}\operatorname{rot}(u_t \vec{H}^3) = 0, \tag{5.4}$$

$$(j = 1,2,3:) \quad \operatorname{div} h^j = 0, \tag{5.5}$$

with boundary conditions

$$u = \Delta u = 0, \quad \nu \times h^j = 0, \quad j = 1,2,3, \tag{5.6}$$

and initial conditions

$$u(\cdot,0) = u_0, \quad u_t(\cdot,0) = u_1, \quad h^j(\cdot,0) = h^j_0, \quad j = 1,2,3. \tag{5.7}$$

In analogy to the two-dimensional case treated in section 4 define the first- and second-order energy terms for a sufficiently smooth solution by

$$\mathcal{E}_1(t) := \frac{d}{\rho_0}\|\Delta u(t,\cdot)\|^2 + \|u_t(t,\cdot)\|^2 + \sum_{k=1}^3 \frac{\alpha_k \rho_k}{\beta_k \rho_0}\|h^k(t,\cdot)\|^2$$

and, with

$$\mathcal{E}_1(t) \equiv \mathcal{E}_1(u, h^1, h^2; t),$$

$$\mathcal{E}_2(t) := \mathcal{E}_1(u_t, h^1_t, h^2_t; t) = \frac{d}{\rho_0}\|\Delta u_t(t,\cdot)\|^2 + \|u_{tt}(t,\cdot)\|^2 + \sum_{k=1}^3 \frac{\alpha_k \rho_k}{\beta_k \rho_0}\|h^k_t(t,\cdot)\|^2.$$

For the sum

$$\mathcal{E}_{nd}(t) := \mathcal{E}_1(t) + \mathcal{E}_2(t) \tag{5.8}$$

the exponential decay will be proved if the unit vectors $\vec{H}^1$, $\vec{H}^2$ and $\vec{H}^3$ are not linearly dependent, i.e. if we have *three essentially different directions* of these magnetic field vectors. Let $D_3 = \det\left(\vec{H}^1\ \vec{H}^2\ \vec{H}^3\right)$ again.

Theorem 5.1. *There exist $K_3 > 0$ and $\kappa_3 > 0$ such that for the solution to (5.1)-(5.7) and all $t \geq 0$*

$$\mathcal{E}_{nd}(t) \leq K_3 \mathcal{E}_{nd}(0) \mathrm{e}^{-\kappa_3 t}$$

holds. $\kappa_3 = \kappa_3(D_3) = \mathcal{O}\left(|D_3|^2\right)$ *as* $D_3 \to 0$.

The decay rate κ_3 vanishes if $\vec{H}^1$, $\vec{H}^2$ and $\vec{H}^3$ become linearly dependent parallel, and it is strongest if they are orthogonal to each other. Remember that $|D_3|$ measures the volume of the parallelepiped spanned by these vectors.

Proof of Theorem 5.1: The proof follows the lines of the proof of Theorem 4.2, we hence only point out the essential modifications. Using essentially the same multipliers, the first modification arises in computing ∇u_t from known $f^k := \operatorname{rot}(u_t \vec{H}k)$, $k = 1, 2, 3$. We get the formula being analogous to (4.16) with the ansatz for a pointwise representation

$$\nabla u_t(t,x) = \sum_{k=1}^{3} \gamma_k(t,x) \vec{H}^k$$

and then computing the coefficients γ_k as

$$\gamma_1 = \frac{\langle f^2, \vec{H}^3 \rangle_{\mathbb{R}^{\nVdash}}}{D_3}, \qquad \gamma_2 = \frac{\langle f^3, \vec{H}^1 \rangle_{\mathbb{R}^{\nVdash}}}{D_3}, \qquad \gamma_3 = \frac{\langle f^1, \vec{H}^2 \rangle_{\mathbb{R}^{\nVdash}}}{D_3}.$$

Thus the analogue to (4.17) now reads

$$c\left(\|u_t\|^2 + \|\nabla u_t\|^2\right) \leq \frac{1}{|D_3|^2}\left(\|\operatorname{rot}(u_t \vec{H}^1)\|^2 + \|\operatorname{rot}(u_t \vec{H}^2)\|^2 + \|\operatorname{rot}(u_t \vec{H}^3)\|^2\right). \tag{5.9}$$

The second modificationa comes up in calculating Δu_t in terms of h_t^j and $\operatorname{rot}\operatorname{rot} h^j$, $j = 1, 2, 3$, starting in two dimensions in (4.22).Now we have

$$f^j := \operatorname{rot}\operatorname{rot}(u_t \vec{H}^j), \tag{5.10}$$

the linear system

$$\mathbb{H}_3 \begin{pmatrix} \partial_1^2 u_t \\ \partial_1 \partial_2 u_t \\ \partial_1 \partial_3 u_t \\ \vdots \\ \partial_3 \partial_2 u_t \\ \partial_3^2 u_t \end{pmatrix} = \begin{pmatrix} f_1^1 \\ f_2^1 \\ f_3^1 \\ \vdots \\ f_2^3 \\ f_3^3 \end{pmatrix}, \tag{5.11}$$

with the matrix $\mathbb{H}_3$ from (3.4) satisfying

$$\det \mathbb{H}_3 = -2(D_3)^3.$$

Exemplarily we compute $\partial_1^2 u_t$ by Cramer's rule as

$$\partial_1^2 u_t \quad = \quad \frac{1}{\det \mathbb{H}_3} \det(\mathcal{H}),$$

where

$$\mathcal{H} := \begin{pmatrix} f_1^1 & -H_2^1 & -H_3^1 & 0 & H_1^1 & 0 & 0 & 0 & H_1^1 \\ f_2^1 & 0 & 0 & -H_1^1 & 0 & -H_3^1 & 0 & 0 & H_2^1 \\ f_3^1 & 0 & 0 & 0 & H_3^1 & 0 & -H_1^1 & -H_2^1 & 0 \\ f_1^2 & -H_1^2 & -H_3^2 & 0 & H_1^2 & 0 & 0 & 0 & H_1^2 \\ f_2^2 & 0 & 0 & -H_1^2 & 0 & -H_3^2 & 0 & 0 & H_2^2 \\ f_3^2 & 0 & 0 & 0 & H_3^2 & 0 & -H_1^2 & -H_2^2 & 0 \\ f_1^3 & -H_2^3 & -H_3^3 & 0 & H_1^3 & 0 & 0 & 0 & H_1^3 \\ f_2^3 & 0 & 0 & -H_1^3 & 0 & -H_3^3 & 0 & 0 & H_2^3 \\ f_3^3 & 0 & 0 & 0 & H_3^3 & 0 & -H_1^3 & -H_2^3 & 0 \end{pmatrix}.$$

Since

$$\begin{aligned} \det(\mathcal{H}) \;=\; D_3^2\; & \big[(H_2^2H_3^3 - H_3^2H_2^3)f_1^1 + (H_1^2H_3^3 - H_3^2H_1^3)f_2^1 + (H_2^2H_1^3 - H_2^3H_1^2)f_3^1 \\ & +(H_3^1H_2^3 - H_2^1H_3^3)f_1^2 + (H_1^1H_3^3 - H_3^1H_1^3)f_2^2 + (H_1^1H_2^3 - H_2^1H_1^3)f_3^2 \\ & +(H_2^1H_3^2 - H_3^1H_2^2)f_1^3 + (H_1^1H_3^2 - H_3^1H_1^2)f_2^3 + (H_2^1H_1^2 - H_1^1H_2^2)f_3^3\big] \end{aligned}$$

(again checked by Maple$^{©}$) we conclude

$$\partial_1^2 u_t == \left(\sum_{k,m=1}^{3} a_{km}^1 f_m^k\right). \tag{5.12}$$

with constants $a_{km}^1 \in \{H_j^i H_l^p \,|\, i,j,p,l = 1,2,3\}$; similarly for $\partial_2^2 u_t$ and $\partial_3^2 u_t$.

This way we obtain the relation (cp. (4.25))

$$D_3 \Delta u_t = \sum_{k,m=1}^{3} \left(\sum_{j=1}^{3} a_{km}^j\right) f_m^k. \tag{5.13}$$

Now we may carry over the remaining arguments from section 4 and thus finish the proof of Theorem 5.1. □

We have the corresponding corollary for the first-order energy term as in section 4.

Corollary 5.2. *There exists $\tilde{K}_3 > 0$ such that for the solution to (5.1) –(5.7) and all $t \geq 0$*

$$\mathcal{E}_1(t) \leq \tilde{K}_3 \mathcal{E}_1(0) e^{-\kappa_3 t}$$

holds.

6 Strong stability for less than n magnetic fields

The results on exponential stability given in Sections 4 and 5, with having an estimated decay rate going to zero as the vectors H^1, H^2 $[H^3]$) tend to become linearly dependent, might be seen as an indication – of course not a proof – that there is no exponential stability given in the case where there are less than n linearly independent magnetic fields, i.e. for $n = 2$ only one magnetic field, and for $n = 3$ either only one or at most two linearly independent magnetic fields. The property of strong stability is proven to be true now exemplarily for the rectangle $\Omega_2 = (0,\pi)^2$ in two dimensions, resp. for the cube $\Omega_3 = (0,\pi)^3$ in three dimensions.

Remark 6.1. *In the previous sections we had assumed, for simplicity, that Ω is smoothly bounded. Although the classical elliptic regularity results for smoothly bounded domains do not carry over to general domains with corners, cf. [2, 6, 7], in particular [7, Example 9.29], the necessary results here remain valid for a square resp. a cube, where we can get the usual elliptic H^2- resp. H^4-regularity results for $-\Delta$ resp. Δ^2 with the boundary conditions used.*

We consider the following systems in this section. For $\underline{n=2}$:

$$\rho_0 u_{tt} + d\Delta^2 u - \alpha_1(\operatorname{rot}\operatorname{rot} h^1)\cdot \vec{H}^1 = 0 \quad \text{in} \quad \Omega_2\times[0,\infty), \tag{6.1}$$

$$\rho_1 h^1_t + \operatorname{rot}\operatorname{rot} h^1 + \beta_1 \operatorname{rot}\operatorname{rot}(u_t\vec{H}^1) = 0 \quad \text{in} \quad \Omega_2\times[0,\infty), \tag{6.2}$$

$$\operatorname{div} h^1 = 0 \quad \text{in} \quad \Omega_2\times[0,\infty), \tag{6.3}$$

with boundary conditions

$$u = \Delta u = 0, \quad \nu\times h^1 = 0, \tag{6.4}$$

and initial conditions

$$u(\cdot,0) = u_0, \quad u_t(\cdot,0) = u_1, \quad h^1(\cdot,0) = h^1_0. \tag{6.5}$$

For $\underline{n=3}$ with one magnetic field:

$$\rho_0 u_{tt} + d\Delta^2 u - \alpha_1 \operatorname{rot}\operatorname{rot} h^1\cdot \vec{H}^1 = 0 \quad \text{in} \quad \Omega_3\times[0,\infty), \tag{6.6}$$

$$\rho_1 h^1_t + \operatorname{rot}\operatorname{rot} h^1 + \beta_1 \operatorname{rot}\operatorname{rot}(u_t\vec{H}^1) = 0 \quad \text{in} \quad \Omega_3\times[0,\infty), \tag{6.7}$$

$$\operatorname{div} h^1 = 0 \quad \text{in} \quad \Omega_3\times[0,\infty), \tag{6.8}$$

with boundary conditions

$$u = \Delta u = 0, \quad \nu\times h^1 = 0, \tag{6.9}$$

and initial conditions

$$u(\cdot,0) = u_0, \quad u_t(\cdot,0) = u_1, \quad h^1(\cdot,0) = h^1_0. \tag{6.10}$$

For $\underline{n=3}$ with two magnetic fields:

$$\rho_0 u_{tt} + d\Delta^2 u - \alpha_1 \operatorname{rot}\operatorname{rot} h^1\cdot \vec{H}^1 - \alpha_2 \operatorname{rot}\operatorname{rot} h^2\cdot\vec{H}^2 = 0 \quad \text{in} \quad \Omega_3\times[0,\infty), \tag{6.11}$$

$$\rho_1 h^1_t + \operatorname{rot}\operatorname{rot} h^1 + \beta_1 \operatorname{rot}\operatorname{rot}(u_t\vec{H}^1) = 0 \quad \text{in} \quad \Omega_3\times[0,\infty), \tag{6.12}$$

$$\rho_2 h^2_t + \operatorname{rot}\operatorname{rot} h^2 + \beta_2 \operatorname{rot}\operatorname{rot}(u_t\vec{H}^2) = 0 \quad \text{in} \quad \Omega_3\times[0,\infty), \tag{6.13}$$

$$(j=1,2:) \quad \operatorname{div} h^j = 0 \quad \text{in} \quad \Omega_3\times[0,\infty), \tag{6.14}$$

with boundary conditions

$$u = \Delta u = 0, \quad \nu\times h^j = 0, \quad j = 1,2, \tag{6.15}$$

and initial conditions

$$u(\cdot,0)=u_0,\quad u_t(\cdot,0)=u_1,\quad h^j(\cdot,0)=h_0^j,\quad j=1,2. \tag{6.16}$$

Then we have the strong stability result for the square resp. the cube in the following

Theorem 6.2. *In both dimensions, $n=2,3$, the semigroups $\left(\mathrm{e}^{\,t\mathcal{A}}\right)_{t\geq 0}$ associated to the problems* (6.1)-(6.5) *resp.* (6.6)-(6.10) *and* (6.11)-(6.16) *are strongly stable, i.e. we have for any initial data $V_0\in\mathcal{H}$*

$$\lim_{t\to\infty}\|\mathrm{e}^{\,t\mathcal{A}}V^0\|_{\mathcal{H}}=0.$$

Proof: Since $\mathcal{A}^{-1}$ is compact, cf. Remark 2.1, we have just to exclude purely imaginary eigenvalues. In the following we assume w.l.o.g. that all constants in the differential equations above are equal to one, i.e.

$$\rho_0=\rho_1=\rho_2=\rho_3=d=\alpha_1=\alpha_2=\alpha_3=\beta_1=\beta_2=\beta_3=1.$$

First we consider $\underline{n=2}$. i.e. system (6.1)-(6.5). Assume that $0\neq V=(u,v,h^1)\in D(\mathcal{A})$ is an eigenvector to the purely imaginary eigenvalue $i\lambda$ with $0\neq\lambda\in\mathbb{R}$, then

$$i\lambda V-\mathcal{A}V=0, \tag{6.17}$$

or, equivalently,

$$\begin{aligned} i\lambda u-v &= 0, && (6.18)\\ i\lambda v+\Delta^2u-(\,\mathrm{rot}\ \mathrm{rot}\,h^1)\cdot\vec{H}^1 &= 0, && (6.19)\\ i\lambda h^1+\,\mathrm{rot}\ \mathrm{rot}\,(v\vec{H}^1)+\,\mathrm{rot}\ \mathrm{rot}\,h^1 &= 0. && (6.20)\end{aligned}$$

Since $\mathrm{Re}\,(\mathcal{A}V,V)_{\mathcal{H}}=0$, we conclude from the dissipation equality (2.5) that $h^1=0$, thus, using (6.18), (6.19), implying

$$\begin{aligned} \Delta^2u &= \lambda^2u, && (6.21)\\ \mathrm{rot}\ \mathrm{rot}\,(u\vec{H}^1) &= 0. && (6.22)\end{aligned}$$

Hence, u is an eigenvector for the biharmonic operator Δ^2 with the boundary conditions $u=\Delta u=0$ in Ω_2, having to satisfy equation (6.22). Thus,

$$u=u(x_1,x_2)=\sin(\gamma_1x_1)\,\sin(\gamma_2x_2), \tag{6.23}$$

with some $(\gamma_1,\gamma_2)\in\mathbb{N}^2$ with $\lambda^2=(\gamma_1^2+\gamma_2^2)^2$. The side condition (6.22) yields

$$\begin{aligned}\begin{pmatrix}0\\0\end{pmatrix} &= \begin{pmatrix}\partial_2\partial_1uH_2^1-\partial_2^2uH_1^1\\ -\partial_1^2uH_2^1+\partial_2\partial_1uH_1^1\end{pmatrix}\\ &= \begin{pmatrix}\gamma_2\gamma_1\cos(\gamma_1x_1)\,\cos(\gamma_2x_2)H_2^1+\gamma_2^2\sin(\gamma_1x_1)\,\sin(\gamma_2x_2)H_1^1\\ \gamma_1^2\sin(\gamma_1x_1)\,\sin(\gamma_2x_2)H_2^1+\gamma_2\gamma_1\cos(\gamma_1x_1)\,\cos(\gamma_2x_2)H_1^1\end{pmatrix}. \qquad (6.24)\end{aligned}$$

But it is easy to see that the latter cannot hold simultaneously for all $(x_1, x_2) \in \Omega_2$. Hence, the eigenvector V cannot exist, and we have proved the strong stability for $n = 2$.

Regarding the case $\underline{n = 3}$ with one magnetic field, i.e. system (6.6)-(6.10), we argue similarly. If V is again an eigenvector to a purely imaginary eigenvalue $i\lambda$, we obtain, as in two dimensions above, that u has to satisfy

$$\Delta^2 u = \lambda^2 u, \tag{6.25}$$

$$\text{rot rot}\,(u\vec{H}^1) = 0. \tag{6.26}$$

Hence, u is again an eigenvector for the biharmonic operator Δ^2 with the boundary conditions $u = \Delta u = 0$ in Ω_3, having to satisfy equation (6.26). Thus,

$$u = u(x_1, x_2, x_3) = \sin(\gamma_1 x_1)\,\sin(\gamma_2 x_2)\,\sin(\gamma_3 x_3), \tag{6.27}$$

with some $(\gamma_1, \gamma_2, \gamma_3) \in \mathbb{N}^3$ with $\lambda^2 = (\gamma_1^2 + \gamma_2^2 + \gamma_3^2)^2$. The side condition (6.26) yields

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \partial_1\partial_2 u H_2^1 - \partial_2^2 u H_1^1 \partial_3^2 u H_1^1 + \partial_1\partial_3 u H_3^1 \\ \partial_2\partial_3 u H_3^1 - \partial_3^2 u H_2^1 - \partial_1^2 u H_2^1 + \partial_2\partial_1 u H_1^1 \\ \partial_3\partial_1 u H_1^1 - \partial_1^2 u H_3^1 - \partial_2^2 u H_3^1 + \partial_3\partial_2 u H_2^1. \end{pmatrix}$$

$$= \begin{pmatrix} \gamma_1\gamma_2 \cos(\gamma_1 x_1)\,\cos(\gamma_2 x_2)\,\sin(\gamma_3 x_3) H_2^1 + (\gamma_2^2 + \gamma_3^2)\sin(\gamma_1 x_1)\,\sin(\gamma_2 x_2)\,\sin(\gamma_3 x_3) H_1^1 \\ \gamma_2\gamma_3 \sin(\gamma_1 x_1)\,\cos(\gamma_2 x_2)\,\cos(\gamma_3 x_3) H_2^1 + (\gamma_1^2 + \gamma_3^2)\sin(\gamma_1 x_1)\,\sin(\gamma_2 x_2)\,\sin(\gamma_3 x_3) H_2^1 \\ \gamma_1\gamma_3 \cos(\gamma_1 x_1)\,\sin(\gamma_2 x_2)\,\cos(\gamma_3 x_3) H_1^1 + (\gamma_1^2 + \gamma_2^2)\sin(\gamma_1 x_1)\,\sin(\gamma_2 x_2)\,\sin(\gamma_3 x_3) H_3^1 \end{pmatrix}$$

$$+ \begin{pmatrix} \gamma_1\gamma_3 \cos(\gamma_1 x_1)\,\sin(\gamma_2 x_2)\,\cos(\gamma_3 x_3) H_3^1 \\ \gamma_1\gamma_2 \cos(\gamma_1 x_1)\,\cos(\gamma_2 x_2)\,\sin(\gamma_3 x_3) H_1^1 \\ \gamma_2\gamma_3 \sin(\gamma_1 x_1)\,\cos(\gamma_2 x_2)\,\cos(\gamma_3 x_3) H_2^1 \end{pmatrix}. \tag{6.28}$$

But the latter cannot hold simultaneously for all $(x_1, x_2, x_3) \in \Omega_3$, take for visualization for example $\vec{H}^1 = (1, 0, 0)$. Hence, the eigenvector V cannot exist, and we have proved the strong stability for $n = 3$ with one magnetic field.

Finally, for $\underline{n = 3}$ with two magnetic fields, i.e. system (6.11)-(6.16), the difference to the case of one magnetic field is that u as in (6.27) has to satisfy (6.26) and *additionally*

$$\text{rot rot}\,(u\vec{H}^2) = 0. \tag{6.29}$$

The more it is impossible that an eigenvector V, as assumed, exist. Thus the strong stability in this case is also proved. □

The question of non-exponentially stability remains open. The conjecture is that for less than n magnetic fields the system is *not* exponentially stable.

Acknowledgements: The second author thanks the National Laboratory for Scientific Computation (LNCC) in Petrópolis for local support during the preparation of this work.

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