WELL-POSEDNESS OF SOLUTIONS FOR MULTI-D HYPERBOLIZED COMPRESSIBLE NAVIER-STOKES EQUATIONS

YUXI HU AND REINHARD RACKE

ABSTRACT. A hyperbolized compressible Navier-Stokes system is considered in multi dimensions, where the parts of the stress tensor are fully relaxed. Local existence is obtained first, then global solutions for *small* data, with estimates being uniform in the three relaxation parameters, the latter allowing the prove a weak convergence result to the classical compressible Navier-Stokes system, *globally in time,* as the relaxation parameters tend to zero. Finally, a blow-up result for *large* data is proved.

Keywords: multi-dimensional compressible Navier-Stokes, hyperbolized, relaxation, global small solution, blow-up for large data

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1. INTRODUCTION

The basic equations of fluid dynamics for $(x,t)\in\mathbb{R}^n\times(0,\infty)$are given by mass conservation, momentum balance and energy conservation as follows:

$$\begin{cases}\partial_t\rho+\operatorname{div}(\rho u)=0,\\ \partial_t(\rho u)+\operatorname{div}(\rho u\otimes u)+\nabla p=\operatorname{div}(S),\\ \partial_t(\rho\mathcal{E})+\operatorname{div}(\rho u\mathcal{E}+pu+q+Su)=0.\end{cases}\tag{1.1}$$

Here, the functions $\rho,u,\mathcal{E},p,S,q$ denote the fluid density, velocity, total energy per unit mass, pressure, stress tensor and heat flux, respectively. The equations for stress S and heat flux q should be given to make the system (1.1) closed. For classical fluid dynamics, the constitutive law for S is given by

$$S=\mu(\nabla u+\nabla u^T-\frac{2}{n}\operatorname{div}uI_n)+\lambda\operatorname{div}uI_n\tag{1.2}$$

and for the heat conduction by Fourier's law,

$$q=-\kappa(\theta)\nabla\theta.\tag{1.3}$$

Here, μ and λ are shear and bulk viscosity coefficients, respectively, which are assumed to be constants. I_n denotes the identity matrix in $\mathbb{R}^n$ and $\kappa(\theta)$ denotes the heat conduction coefficient.

The system (1.1)-(1.3) is called compressible Navier-Stokes-Fourier system, which plays a crucial rule both in physics and mathematics. However, due to its hyperbolic-parabolic structure, the system implies an inherent infinite propagation speed of signals, which contradicts our common sense. Lots of physicists and mathematicians are devoted to establish new models to remove such paradox, see Müller [28], Ruggeri [34] and Boyaval [1]. The common feature of these works are replacing the constitutive relations (1.2) and (1.3) by various kinds of relaxed equations. Thus, the diffusive hyperbolic-parabolic coupled equations are reduced to pure hyperbolic equations with damping terms which indicates the property of finite propagation speed of signals.

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On the other hand, for some complex fluids, like macromolecular or polymeric fluids, S satisfies the following constitutive equation

$$\tau\dot{S}+S=\mu(\nabla u+\nabla u^T-\frac{2}{n}\mathrm{div}uI_n)+\lambda\mathrm{div}uI_n \tag{1.4}$$

where $\dot{S}=\partial_t S+u\cdot\nabla S$. The equation (1.4) is obtained by combining Newton's law of viscosity and Hooke's law of elasticity. The positive parameter τ is the relaxation time describing the time lag in the response of the stress tensor to velocity gradient. A fluid obeying equation (1.4) is called Maxwell flow, see [29]. Replacing the material derivative $\dot{S}$ by objective derivative, we obtain an objective constitutive equation of compressible Oldroyd-B type models:

$$\tau\mathring{S}+S=\mu(\nabla u+\nabla u^T-\frac{2}{n}\mathrm{div}uI_n)+\lambda\mathrm{div}uI_n \tag{1.5}$$

where $\mathring{S}:=\dot{S}+SW(u)-W(u)S-a(SD(u)+D(u)S)$ with $W(u)=\frac{1}{2}(\nabla u-\nabla u^T)$, $D(u)=\frac{1}{2}(\nabla u+\nabla u^T)$ and $-1\le a\le 1$ be a constant. We should mention that, even for a simple fluid, the effect of relaxation can not always be neglected, see [31] with the experiments of high-frequency (20GHZ) vibration of nanoscale mechanical devices immersed in water-glycerol mixtures. In 2014, Yong [39] proposed a new model in which he divided the stress tensor S into $S_1+S_2I_n$ and did relaxation for S_1 and S_2 in the following form

$$\hat{\tau}_1\partial_t S_1+S_1=\mu(\nabla u+\nabla u^T-\frac{2}{n}\mathrm{div}uI_n), \tag{1.6}$$

$$\hat{\tau}_2\partial_t S_2+S_2=\lambda\mathrm{div}u. \tag{1.7}$$

Here $\hat{\tau}_1$ and $\hat{\tau}_2$ are called shear and bulk relaxation time, respectively. The constitutive equations (1.6)-(1.7) are called revised Maxwell's law. Note that even though there is no quadratic terms, like $u\cdot\nabla S_1$ and $u\cdot\nabla S_2$ in (1.6) resp. (1.7), and thus does not have the property of Galilean invariance, let alone objective, this model is shown to be the best one in comparison to various other models for the dynamic behavior of linear viscoelasticity flows considered in [3]. Indeed, for a compressible viscoelastic fluid, Chakraborty and Sader [3] show that this model provides a general formalism with which to characterize the fluid-structure interaction of nanoscale mechanical devices vibrating in simple liquids.

In quite a similar way, going back to Cattaneo [2], Christov [6] proposed the following objective constitutive equation for q,

$$\hat{\tau}_0\left(\partial_t q+u\cdot\nabla q-q\cdot\nabla u+(\mathrm{div}u)q\right)+q+\kappa(\theta)\nabla\theta=0, \tag{1.8}$$

which gives rise to heat waves with finite propagation speed. Here $\hat{\tau}_0>0$ is the relaxation time.

We note that the system (1.1) with two objective constitutive relations (1.5) and (1.8) is yet too complicated to analyze or simulate. Actually, only in the 1-D case, the authors [18] obtained a local and global well-posedness theorem for some small initial data. The h-D case is mathematically not yet accessible, even locally, because of the loss of symmetric structure for first-order quasilinear system. So, in practice, the model (1.1), (1.5) and (1.8) is always modified with an additional viscosity (called "a retardation time"). However, such a modification spoils the structure of hyperbolicity and therefore loses the property of finite speed.

In this paper, we shall consider the following constitutive equations:

$$\tau_0(\theta)\rho(\partial_t q+u\cdot\nabla q)+q+\kappa(\theta)\nabla\theta=0, \tag{1.9}$$

and

$$\tau_1\rho\partial_t(S_1+u\cdot\nabla S_1)+S_1=\mu(\nabla u+\nabla u^T-\frac{2}{n}\mathrm{div}uI_n), \tag{1.10}$$

$$\tau_2\rho\partial_t(S_2+u\cdot\nabla S_2)+S_2=\lambda\mathrm{div}u, \tag{1.11}$$

with $S = S_1 + S_2 I_n$. The constitutive equation (1.10)-(1.11) was proposed by Freistühler [10, 11] for the isentropic case, see also Ruggeri [34] and Müller [28] for a similar model in the non-isentropic case. Formally, by replacing $\partial_t S$ by $\dot{S}$ and take $\hat{\tau}_1 = \tau_1 \rho$ and $\hat{\tau}_2 = \tau_2 \rho$ in (1.6)-(1.7), one can also get (1.10)-(1.11). Similarly, by neglecting the quadratic terms $q \cdot \nabla u$ and $(\text{div} u)q$ and taking $\hat{\tau}_0 = \tau_0(\theta)\rho$ in (1.8), we get (1.9) immediately.

Furthermore, we assume that the specific total Energy is given by

$$\mathcal{E} = \frac{1}{2}u^2 + \frac{\tau_1}{4\mu}S_1^2 + \frac{\tau_2}{2\lambda}S_2^2 + e, \tag{1.12}$$

and the specific internal energy e and the pressure p are given by

$$e = C_v \theta + a(\theta) q^2, \quad p = R\rho\theta, \tag{1.13}$$

where $a(\theta) = \frac{Z(\theta)}{\theta} - \frac{1}{2}Z'(\theta)$ and $Z(\theta) = \frac{\tau_0(\theta)}{\kappa(\theta)}$. C_v, R denote the heat capacity at constant volume and the gas constant, respectively. p and e satisfy the usual thermodynamic equation

$$\rho^2 e_\rho = p - \theta p_\theta.$$

Here, we note that, because of relaxation, the specific energy are composed of three parts: $\frac{1}{2}u^2$ be kinetic energy, $\frac{\tau_1}{4\mu}S_1^2 + \frac{\tau_2}{2\lambda}S_2^2$ the relaxed potential energy and e the specific internal energy.

The dependence of the internal energy on q^2 and the structure of the dependence of a, Z on θ is indicated by Coleman et al. in [7], where they rigorously prove that for heat equations with Cattaneo-type law, the formulation (1.13) is consistent with the second law of thermodynamics, see also [4, 8, 37].

Note that for the case $\tau_0 = \tau_1 = \tau_2 = 0$, the system (1.1), (1.9)-(1.13) reduces to the classical compressible Navier-Stokes-Fourier equations, for which there are lots of results concerning the local and global well-posedness of strong and/or weak solutions. In particular, the local existence and uniqueness of smooth solutions was established by Serrin [35] and Nash [30] for initial data far away from vacuum. Later, Matsumura and Nishida [27] got global smooth solutions for small initial data without vacuum. For large data, Xin [38], Cho and Jin [5] showed that smooth solutions must blow up in finite time if the initial data has a vacuum state. See [13, 14, 25, 22, 23, 9] for global existence of weak solutions. Neglecting the quadratic nonlinear terms $u \cdot \nabla q$ and $u \cdot \nabla S_i, i = 1, 2$ in (1.9) resp. $(1.10) - (1.11)$, the cases $\tau_0 > 0, \tau_1 = \tau_2 = 0$ (Cattaneo's law) and $\tau_0 = 0, \tau_1 > 0, \tau_2 > 0$ (revised Maxwell's law) have been studied in $\mathbb{R}^n$, $n \geq 2$, respectively, in our papers [16, 17]. For the *one-dimensional* case, we had considered the relaxation both for q and S with Galilean invariance (also objective) form. In [18], we showed the global existence of smooth solutions with small initial data and convergence to classical system as relaxation parameter goes to zero. In our paper with Wang [21], a blow-up result for large data was shown, hereby also yielding an interesting example, where the relaxed and the non-relaxed system are close to each other on the linearized level, globally for small data for the nonlinear system, and on any finite time horizon for the nonlinear one, but differ qualitatively for large data. The results in [21] were extended by the authors in [20] to a related model, where smallness assumptions to assure hyperbolicity were no longer needed.

Recently, for the *multi-dimensional case*, the authors [19] have considered the case $\tau_0 > 0, \tau_1 = 0, \tau_2 > 0$ and obtained a global small smooth solutions for $\mu > 0$ and a blow-up result with some large data for $\mu = 0$. To our knowledge, most of the previous results only concern partial relaxations, for which regularity of elliptic operators plays an important role in getting both local and global solutions. It leaves the open question whether the system (1.1) with full relaxation (the case $\tau_0 > 0, \tau_1 > 0, \tau_2 > 0$ in (1.9)-(1.11)) is well-posed either locally or globally. We shall solve this problem in this paper.

We investigate the Cauchy problem to system (1.1), (1.9)-(1.13) for the functions

$$(\rho, u, \theta, q, S_1, S_2) : \mathbb{R}^n \times [0, +\infty) \to (0, \infty) \times \mathbb{R}^n \times (0, \infty) \times \mathbb{R}^n \times \mathbb{R}^{n^2} \times \mathbb{R}$$

with initial condition

$$(\rho(x,0),u(x,0),\theta(x,0),q(x,0),S_1(x,0),S_2(x,0))=(\rho_0,u_0,\theta_0,q_0,S_{01},S_{02}). \tag{1.14}$$

We need the following assumptions for the coefficients $\tau_0(\theta)$ and $\kappa(\theta)$.

Assumption 1.1. *Let $\tau_0(\theta)=\tau_0\cdot\tilde{\tau}_0(\theta)$ with $\tau_0>0$ be any fixed constant. $\tilde{\tau}_0(\theta)$ and $\kappa(\theta)$ are assumed to be smooth functions of θ satisfying*

$$\tilde{\tau}_0(\theta)>0, \quad \kappa(\theta)>0, \quad \textit{for all } \theta>0.$$

Assumption 1.2. *Let*

$$a(\theta)>0, \quad a'(\theta)\geq 0, \quad \frac{1}{2}\left(\frac{Z(\theta)}{\theta}\right)'\geq 0, \quad \textit{for all } \theta>0. \tag{1.15}$$

The inequality $a'(\theta)\geq 0$ implies $e_\theta\geq C_v>0$, which makes the system (1.1)-(1.3) uniformly hyperbolic without smallness condition. The third inequality in (1.15) will give the L^2-estimates for q from Remark 3.3 below, which will be used in the blow-up result. Note also that by choosing $Z(\theta)=\frac{\tau_0(\theta)}{\kappa(\theta)}=k\theta^\alpha$ with k being any constant and $1\leq\alpha<2$, the assumption (1.15) holds.

In the following, we consider w.l.o.g. the *three-dimensonal case*, that is, $n=3$. Our first main result is stated as follows.

Theorem 1.3. *(Uniform global existence). Let Assumption 1.1 hold and trace$(S_{01})=0$. There exists a sufficiently small constant $\delta>0$ such that, if*

$$\|(\rho_0-1,u_0,\theta_0-1,\sqrt{\tau_0}q_0,\sqrt{\tau_1}S_{01},\sqrt{\tau_2}S_{02})\|_{H^3}<\delta,$$

then the initial value problem (1.1), (1.9)-(1.14) *has a unique solution $(\rho,u,\theta,q,S_1,S_2)$ globally in time such that*

$$(\rho-1,u,\theta-1,q,S_1,S_2)\in C([0,\infty);H^3),$$

and

$$(\nabla\rho,\nabla u,\nabla\theta)\in L^2((0,\infty);H^2), \quad (q,S_1,S_2)\in L^2((0,\infty);H^3).$$

Moreover, for any $t>0$ we have

$$\|(\rho-1,u,\theta-1,\sqrt{\tau_0}q,\sqrt{\tau_1}S_1,\sqrt{\tau_2}S_2)\|_{H^3}^2+\int_0^t\left(\|(\nabla\rho,\nabla u,\nabla\theta)\|_{H^2}^2+\|(q,S_1,S_2)\|_{H^3}^2\right)\mathrm{d}t\leq CE_0^2, \tag{1.16}$$

where C is a constant being independent of t,τ_0,τ_1,τ_2 and of the initial data. Moreover, the solution decays uniformly in the sense

$$\|\nabla(\rho,u,\theta,\sqrt{\tau_0}q,\sqrt{\tau_1}S_1,\sqrt{\tau_2}S_2)\|_{L^2}\to 0 \quad \text{as}\, t\to\infty. \tag{1.17}$$

Based on the uniform estimates of solutions, we have the following convergence theorem.

Theorem 1.4. *(Global weak convergence). Let $\tau=(\tau_0,\tau_1,\tau_2)$ and $(\rho^\tau,u^\tau,\theta^\tau,q^\tau,S_1^\tau,S_2^\tau)$ be the global solutions obtained in Theorem 1.3, then there exists $(\rho^0,u^0,\theta^0)\in L^\infty((0,\infty);H^3(\mathbb{R}))$ and $(q^0,S_1^0,S_2^0)\in L^2((0,\infty);H^3(\mathbb{R}))$, such that, as $\tau\to 0$, up to subsequences,*

$$(\rho^\tau,u^\tau,\theta^\tau)\longrightarrow(\rho^0,u^0,\theta^0) \quad \text{weak}-* \quad \text{in} \quad L^\infty((0,\infty);H^3(\mathbb{R})), \tag{1.18}$$

$$(q^\tau,S_1^\tau,S_2^\tau)\longrightarrow(q^0,S_1^0,S_2^0) \quad \text{weakly} \quad \text{in} \quad L^2((0,\infty);H^3(\mathbb{R})), \tag{1.19}$$

where (ρ^0,u^0,θ^0) is a distributional solution to the three-dimensional full compressible Navier-Stokes equations (1.1)-(1.3)*. Moreover, for any $T>0$,*

$$(\rho^\tau,u^\tau)\to(\rho^0,u^0) \quad \text{strongly} \quad \text{in} \quad C([0,T],H^2_{loc}(\mathbb{R}^3))$$

and, a.e.,

$$q^0=-\kappa(\theta^0)\nabla\theta^0, S_1^0=\mu(\nabla u^0+(\nabla u^0)^T-\frac{2}{3}\text{div}u^0I_3), S_2^0=\lambda\text{div}u^0.$$

Remark 1.1. *More regularity in $\theta^{(\tau)}$ seems difficult to obtain, though claimed in* [32] *for the 1-d case.*

According to Theorem 1.3, a natural question is that whether the studied system possess a global solution for large initial data. We give a negative answer to this question. Actually, we show that there exists some large initial data for which the classical solutions must blow-up in finite time. Before we state the next theorem, it is helpful to define some useful averaged quantities, extending [36]:

$$F(t) := \int \rho u \cdot x \mathrm{d}x - 3\tau_2 \int \rho S_2 \mathrm{d}x, \tag{1.20}$$

$$G(t) := \int (\rho \mathcal{E}(x,t) - \bar{\rho}\bar{\mathcal{E}}) \mathrm{d}x, \tag{1.21}$$

with $\bar{\rho} = 1$ and $\bar{\mathcal{E}} = C_v$. Our blow-up result is stated as follows.

Theorem 1.5. *Let Assumptions 1.1 and 1.2 hold. Let the initial data be given as in Theorem 2.1 and Proposition 6.1 . Moreover, we assume that*

$$G(0) > 0. \tag{1.22}$$

Then, there exists $(\rho_0, u_0, \theta_0, q_0, S_{01}, S_{02})$ satisfying

$$F(0) > \frac{128\pi\sigma \max \rho_0 M^4}{3(5-3\gamma)} \tag{1.23}$$

and

$$\left(\frac{(5-3\gamma)\mu\tau_1}{M^2} + 3(\gamma - 1) \right) (H_0 + \frac{\max \rho_0}{2} \|u_0\|_{L^2}^2) \leq \frac{256\pi\sigma^2 \max \rho_0 M^3}{3(5-3\gamma)} \tag{1.24}$$

where H_0 is defined in (6.2)*, σ, M are defined in Proposition 6.1, such that the length T_0 of the maximal interval of existence of a smooth solution $(\rho, u, \theta, q, S_1, S_2)$ of* (1.1)*,* (1.9)*-*(1.13) *is finite, provided the compact support of the initial data is sufficiently large and $\gamma := 1 + \frac{R}{C_v}$ is sufficiently close to* 1.

Remark 1.2. *We mention that, for the classical Naviver-Stokes-Fourier system, it is still a famous open problem whether or not there exists a unique global large smooth solution. In this regard, we establish a result which illustrate a possible different qualitative behavior for multi-dimensional relaxed and classical system. This change of qualityative behavior has already been shown in the one dimensional case, where the authors proved that the solutions to the relaxed system blow up for some large data, contrasting the situation of the classical system, see* [21] *for more details.*

Remark 1.3. *In comparison to our previous paper* [19]*, we modified the model and, now, can avoid the smallness assumptions for the blow-up result there.*

The proofs of Theorem 1.3, Theorem 1.4 and of Theorem 1.5, are given in Section 4, 5 and 6, respectively. The main novelties of the paper are

- local existence for the *multi-dimensional case* with fully relaxed stress tensor
- global existence for small data
- *global in time* convergence the classical compressible Navier-Stokes system for relaxation parameters tending to zero
- blow-up for large data for the relaxed system

Finally, we introduce some notation. $W^{m,p} = W^{m,p}(\mathbb{R}^n)$, $0 \leq m \leq \infty$, $1 \leq p \leq \infty$, denotes the usual Sobolev space with norm $\| \cdot \|_{W^{m,p}}$. H^m and L^p stand for $W^{m,2}(\mathbb{R}^n)$ resp. $W^{0,p}(\mathbb{R}^n)$. For $m \times d$-matrices $B = (b_{jk})$, $M = (m_{jk})$, we denote $A : B = \sum_{j=1}^m \sum_{k=1}^d b_{jk} m_{jk}$, and $M^2 = M : M$. For $m \in \mathbb{N}_0$ we denote by $\nabla^m v$ derivatives of v of order m.

2. Local existence

In this section, we show the local existence of smooth solutions for the initial value problem (1.1), (1.9)-(1.14).

Theorem 2.1. *(Local existence) Let $s \geq s_0 + 1$ with $s_0 \geq [\frac{n}{2}] + 1$ be integers. Suppose that the initial data $(\rho_0 - 1, u_0, \theta_0 - 1, q_0, S_{01}, S_{02})$ are in H^s with*

$$\min_{x\in\mathbb{R}^n}(\rho_0(x), \theta_0(x)) > 0, \quad \min_{x\in\mathbb{R}^n}(C_v + a'(\theta_0(x))q_0^2(x)) > 0 \tag{2.1}$$

and $\mathrm{trace}(S_{01}) = 0$. *Then, there exists $T_{ex} > 0$ such that the system* (1.1), (1.9)-(1.14) *has an unique classical solution $(\rho, u, \theta, q, \theta, S_1, S_2)$ satisfying*

$$(\rho - 1, u, \theta - 1, q, S_1, S_2) \in C([0, T_{ex}], H^s) \cap C^1([0, T_{ex}], H^{s-1}), \tag{2.2}$$

and

$$\min_{x\in\mathbb{R}^n}(\rho(t,x), \theta(t,x)) > 0, \quad \min_{x\in\mathbb{R}^n}(C_v + a'(\theta(t,x))q^2(t,x)) > 0 \qquad \forall t \in [0, T_{ex}].$$

Proof. Using the energy equation $(1.1)_3$, we derive the following equation for θ:

$$\rho e_\theta \partial_t \theta + \left(\rho u e_\theta - \frac{2a(\theta)}{Z(\theta)} q\right) \nabla\theta + p\mathrm{div}u + \mathrm{div}q = \frac{2a(\theta)}{\tau_0(\theta)} q^2 + \frac{1}{2\mu} S_1^2 + \frac{1}{\lambda} S_2^2. \tag{2.3}$$

So, we have

$$\begin{cases} \partial_t\rho + \mathrm{div}(\rho u) = 0, \\ \rho\partial_t u + \rho u \cdot \nabla u + \nabla p = \mathrm{div}S_1 + \nabla S_2, \\ \rho e_\theta \partial_t\theta + \left(\rho u e_\theta - \frac{2a(\theta)}{Z(\theta)} q\right)\nabla\theta + p\mathrm{div}u + \mathrm{div}q = \frac{2a(\theta)}{\tau_0(\theta)} q^2 + \frac{1}{2\mu} S_1^2 + \frac{1}{\lambda} S_2^2, \\ \tau_0(\theta)\rho(\partial_t q + u \cdot \nabla q) + q + \kappa(\theta)\nabla\theta = 0, \\ \tau_1\rho(\partial_t S_1 + u \cdot \nabla S_1) + S_1 = \mu(\nabla u + \nabla u^T - \frac{2}{3}\mathrm{div}u I_3), \\ \tau_2\rho(\partial_t S_2 + u \cdot \nabla S_2) + S_2 = \lambda\mathrm{div}u. \end{cases} \tag{2.4}$$

We take the trace on both sides of equation $(2.4)_5$ and denote $\mathrm{trace}(S_1)$ by T_r, then we get

$$\tau_0\rho(\partial_t T_r + u \cdot \nabla T_r) + T_r = 0$$

Since $T_r(0,x) = \mathrm{trace}(S_1)(0,x) = 0$ for all $x \in \mathbb{R}^3$, we get immediately $T_r(t,x) = 0$ for all $(t,x) \in (0,\infty) \times \mathbb{R}^3$. Therefore, without loss of generality, we assume S_1 to take the following form:

$$S_1 = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & -a_{11} - a_{22} \end{pmatrix}. \tag{2.5}$$

Let $\omega = (\rho, u, \theta, q, a_{11}, a_{12}, a_{13}, a_{22}, a_{23}, S_2)$. Then, we have

$$A_0(\omega)\omega_t + \sum_{j=1}^{3} A_j(\omega)\partial_{x_j}\omega + L(\omega)\omega = f(\omega). \tag{2.6}$$

Here,

$$A_0(\omega) = \mathrm{diag}\left\{\frac{p_\rho}{\rho}, \rho, \rho, \rho, \frac{\rho e_\theta}{\theta}, \frac{Z(\theta)}{\theta}\rho, \frac{3\tau_1\rho}{4\mu}, \frac{\tau_1\rho}{\mu}, \frac{\tau_1\rho}{\mu}, \frac{3\tau_1\rho}{4\mu}, \frac{\tau_1\rho}{\mu}, \frac{\tau_2\rho}{\lambda}\right\},$$

$$L(\omega) = \mathrm{diag}\left\{0, 0, 0, 0, 0, \frac{1}{\kappa(\theta)\theta}, \frac{3}{4\mu}, \frac{1}{\mu}, \frac{1}{\mu}, \frac{3}{4\mu}, \frac{1}{\mu}, \frac{1}{\lambda}\right\},$$

$$f(\omega) = \mathrm{diag}\{0, 0, 0, 0, \frac{2a(\theta)}{\tau_0(\theta)} q^2 + \frac{1}{2\mu} S_1^2 + \frac{1}{\lambda} S_2^2, 0, 0, 0, 0, 0, 0\},$$

$$\sum_{j=1}^{3} A_j(\omega)\xi_j = \begin{pmatrix} \frac{p_\rho}{\rho}u\cdot\xi & p_\rho\xi & 0 & 0_{1\times3} & 0_{1\times5} & 0 \\ p_\rho\xi^T & \rho u\cdot\xi I_3 & p_\theta\xi^T & 0_{3\times3} & C_{3\times5}(\xi) & -\xi^T \\ 0 & p_\theta\xi & (\frac{\rho e_\theta}{\theta}u - \frac{2a(\theta)}{\theta Z(\theta)}q)\cdot\xi & \frac{1}{\theta}\xi & 0_{1\times5} & 0 \\ 0_{3\times1} & 0_{3\times3} & \frac{1}{\theta}\xi^T & \frac{Z(\theta)}{\theta}u\cdot\xi I_3 & 0_{3\times5} & 0_{3\times1} \\ 0_{5\times1} & D_{5\times3}(\xi) & 0_{5\times1} & 0_{5\times3} & \tau_1\rho u\cdot\xi E_{5\times5} & 0_{5\times1} \\ 0 & -\xi & 0 & 0_{1\times3} & 0_{1\times5} & \frac{\tau_2}{\lambda}\rho u\cdot\xi \end{pmatrix},$$

where $E_{5\times5} := \mathrm{diag}\{\frac{3}{4\mu}, \frac{1}{\mu}, \frac{1}{\mu}, \frac{3}{4\mu}, \frac{1}{\mu}\}$, and

$$C_{3\times5}(\xi) := \begin{pmatrix} -\xi_1 & -\xi_2 & -\xi_3 & 0 & 0 \\ 0 & -\xi_1 & 0 & -\xi_2 & -\xi_3 \\ \xi_3 & 0 & \xi_3-\xi_1 & 0 & -\xi_2 \end{pmatrix}, D_{5\times3}(\xi) := \begin{pmatrix} -\xi_1 & \frac{\xi_2}{2} & \frac{\xi_3}{2} \\ -\xi_2 & -\xi_1 & 0 \\ -\xi_3 & 0 & -\xi_1 \\ \frac{\xi_1}{2} & -\xi_2 & \frac{\xi_3}{2} \\ 0 & -\xi_3 & -\xi_2 \end{pmatrix}$$

for each $\xi \in \mathbb{S}^2 := \{\xi \in \mathbb{R}^3 \,|\, |\xi| = 1\}$. Note that the matrix $\sum_{j=1}^{3} A_j\xi_j$ is not symmetric. Therefore, the theory of symmetric-hyperbolic-parabolic system does not apply directly. As in [17], we can perform a transformation to overcome this problem. Let $b_{11} := \frac{a_{11}+a_{22}}{2}, b_{22} := \frac{a_{11}-a_{22}}{2}$. This particularly implies $a_{11} = b_{11} + b_{22}, a_{22} = b_{11} - b_{22}$ and S_1 become

$$\tilde{S}_1 = \begin{pmatrix} b_{11}+b_{22} & a_{12} & a_{13} \\ a_{12} & b_{11}-b_{22} & a_{23} \\ a_{13} & a_{23} & -2b_{11} \end{pmatrix}.$$

Let $\tilde{\omega} := (\rho, u, \theta, q, b_{11}, a_{12}, a_{13}, b_{22}, a_{23}, S_2)$. Then system (2.4) can be rewritten as

$$\tilde{A}_0(\tilde{\omega})\tilde{\omega}_t + \sum_{j=1}^{3} \tilde{A}_j(\tilde{\omega})\partial_{x_j}\tilde{\omega} + \tilde{L}(\tilde{\omega})\tilde{\omega} = f(\tilde{\omega}), \tag{2.7}$$

with

$$\tilde{A}_0(\tilde{\omega}) := \mathrm{diag}\left\{\frac{p_\rho}{\rho}, \rho, \rho, \rho, \frac{\rho e_\theta}{\rho}, \frac{Z(\theta)}{\theta}\rho, \frac{3\tau_1}{\mu}, \frac{\tau_1}{\mu}, \frac{\tau_1}{\mu}, \frac{\tau_1}{\mu}, \frac{\tau_1}{\mu}, \frac{\tau_2}{\lambda}\right\},$$

$$\tilde{L}(\tilde{\omega}) := \mathrm{diag}\left\{0, 0, 0, 0, 0, \frac{1}{\kappa(\theta)\theta}, \frac{3}{\mu}, \frac{1}{\mu}, \frac{1}{\mu}, \frac{1}{\mu}, \frac{1}{\mu}, \frac{1}{\lambda}\right\}$$

and

$$\sum_{j=1}^{3} \tilde{A}_j(\tilde{\omega})\xi_j = \begin{pmatrix} \frac{p_\rho}{\rho}u\cdot\xi & p_\rho\xi & 0 & 0_{1\times3} & 0_{1\times5} & 0 \\ p_\rho\xi^T & \rho u\cdot\xi I_3 & p_\theta\xi^T & 0_{3\times3} & \tilde{C}_{3\times5}(\xi) & -\xi^T \\ 0 & p_\theta\xi & (\frac{\rho e_\theta}{\theta}u - \frac{2a(\theta)}{\theta Z(\theta)}q)\cdot\xi & \frac{1}{\theta}\xi & 0_{1\times5} & 0 \\ 0_{3\times1} & 0_{3\times3} & \frac{1}{\theta}\xi^T & \frac{Z(\theta)}{\theta}u\cdot\xi I_3 & 0_{3\times5} & 0_{3\times1} \\ 0_{5\times1} & \tilde{D}_{5\times3}(\xi) & 0_{5\times1} & 0_{5\times3} & \tau_1\rho u\cdot\xi E_{5\times5} & 0_{5\times1} \\ 0 & -\xi & 0 & 0_{1\times3} & 0_{1\times5} & \frac{\tau_2}{\lambda}\rho u\cdot\xi \end{pmatrix},$$

where

$$\tilde{C}_{3\times5}(\xi) := \begin{pmatrix} -\xi_1 & -\xi_2 & -\xi_3 & -\xi_1 & 0 \\ -\xi_2 & -\xi_1 & 0 & \xi_2 & -\xi_3 \\ 2\xi_3 & 0 & -\xi_1 & 0 & -\xi_2 \end{pmatrix}, \tilde{D}_{5\times3}(\xi) := \begin{pmatrix} -\xi_1 & -\xi_2 & 2\xi_3 \\ -\xi_2 & -\xi_1 & 0 \\ -\xi_3 & 0 & -\xi_1 \\ -\xi_1 & \xi_2 & 0 \\ 0 & -\xi_3 & -\xi_2 \end{pmatrix}.$$

Note that $\tilde{C}_{3\times5}(\xi) = \tilde{D}^T_{5\times3}(\xi)$ for each $\xi \in \mathbb{S}^2$. Therefore, the system (2.7) is a symmetric-hyperbolicsystem, and the local existence theorem follows, see [26, 24, 33].

In the two-dimensional case $n = 2$, we only remark that one can easily check that the system can be written in a symmetric form immediately. This is different from the 3-d case, for which we needed further transformations to get a system in a symmetric form. □

Remark 2.1. *Note that the initial assumption* (2.1) *is to ensure the hyperbolicity of system* (2.7). *In fact, e_θ is a smooth function of $\theta - 1, \sqrt{\tau_0}q$. Therefore, for sufficiently small of $\theta - 1$ and $\sqrt{\tau_0}q$ (the situation for global solutions), it can be deduced that $e_\theta \geq \frac{C_v}{2} > 0$.*

3. Entropy equation and some preliminary inequalities

In this part, we first derive an entropy equation for system (1.1), (1.9)-(1.13). We define the entropy η by

$$\eta := C_v \ln\theta - R\ln\rho - \left(\frac{Z(\theta)}{2\theta}\right)' q^2. \tag{3.1}$$

Similar to [18], we have for a local solution

Lemma 3.1.

$$\partial_t(\rho\eta) + \mathrm{div}(\rho u\eta) + \mathrm{div}\left(\frac{q}{\theta}\right) = \frac{q^2}{\kappa(\theta)\theta^2} + \frac{S_1^2}{2\mu\theta} + \frac{S_2^2}{\lambda\theta}. \tag{3.2}$$

Proof. Combining the equations $(1.1)_2, (1.1)_3$ and (1.10)-(1.11) , we derive the equation for e:

$$\rho e_t + \rho u\cdot\nabla e + p\mathrm{div}u + \mathrm{div}q = \frac{1}{2\mu}S_1^2 + \frac{1}{\lambda}S_2^2. \tag{3.3}$$

Recalling (1.13) and dividing equation (3.3) by θ, we have

$$\frac{\rho}{\theta}\partial_t(C_v\theta + a(\theta)q^2) + \frac{\rho}{\theta}u\cdot\nabla(C_v\theta + a(\theta)q^2) + R\rho\mathrm{div}u + \frac{\mathrm{div}q}{\theta} = \frac{1}{2\mu\theta}S_1^2 + \frac{1}{\lambda\theta}S_2^2. \tag{3.4}$$

We calculate

$$\begin{aligned}
&\frac{\rho}{\theta}(a(\theta)q^2)_t + \frac{\rho}{\theta}u\cdot\nabla(a(\theta)q^2)\\
&= \rho\left(\frac{a(\theta)}{\theta}q^2\right)_t + \rho\frac{a(\theta)}{\theta^2}q^2\theta_t + \rho u\cdot\nabla\left(\frac{a(\theta)}{\theta}q^2\right) + \rho\frac{a(\theta)}{\theta^2}q^2 u\cdot\nabla\theta\\
&= \rho\left(\frac{a(\theta)}{\theta}q^2\right)_t - \rho\left(\frac{Z(\theta)}{2\theta^2}q^2\right)_t + \rho\frac{Z(\theta)}{\theta^2}(\frac{1}{2}q^2)_t + \rho u\nabla\left(\frac{a(\theta)}{\theta}q^2\right) - \rho u\nabla\left(\frac{Z(\theta)}{2\theta^2}q^2\right) + \rho u\frac{Z(\theta)}{\theta^2}\cdot\nabla(\frac{1}{2}q^2)\\
&= \rho\left(\left(\frac{a(\theta)}{\theta} - \frac{Z(\theta)}{2\theta^2}\right)q^2\right)_t + \rho u\cdot\nabla\left(\left(\frac{a(\theta)}{\theta} - \frac{Z(\theta)}{2\theta^2}\right)q^2\right) - \frac{1}{\kappa(\theta)\theta^2}q^2 - \frac{1}{\theta^2}\nabla\theta\cdot q,
\end{aligned}$$

where we have used the identity $(\frac{Z(\theta)}{2\theta^2})_t = \frac{a(\theta)}{\theta^2}\theta_t$.

On the other hand, using the mass equation $(1.1)_1$, we have

$$R\rho\mathrm{div}u = -R\rho((\ln\rho)_t + u\cdot\nabla(\ln\rho)_x).$$

Combining the above estimates and noting that

$$\frac{a(\theta)}{\theta} - \frac{Z(\theta)}{2\theta^2} = -\left(\frac{Z(\theta)}{2\theta}\right)',$$

we get the desired results. □

Remark 3.1. *The entropy defined in* (3.1) *satisfies the modified Gibb relation*

$$\theta\mathrm{d}\eta = \mathrm{d}e + p\mathrm{d}V - \frac{Z(\theta)}{\theta}q\mathrm{d}q, \tag{3.5}$$

where $V = \frac{1}{\rho}$ denotes the specific volume per unit mass.

Remark 3.2. *In case $\tau_0 = \tau_1 = \tau_2 = 0$, the entropy η defined in (3.1) is reduced to the quantity $C_v \ln\theta - R\ln\rho$ which is the entropy for classical fluid dynamics. Moreover, the entropy equation (3.2) is reduced to the following classical entropy equation*

$$\partial_t(\rho\eta) + \mathrm{div}(\rho u\eta) - \mathrm{div}(\frac{\kappa\nabla\theta}{\theta}) = \frac{\kappa|\nabla\theta|^2}{\theta^2} + \frac{1}{\theta}\left(\frac{\mu}{2}|\nabla u + \nabla u^T - \frac{2}{n}\mathrm{div}u I_n|^2 + \lambda|\mathrm{div}u|^2\right). \tag{3.6}$$

Remark 3.3. *Combining the mass equation $(1.1)_1$, the energy equation $(1.1)_3$ and the entropy equation (3.2), we get*

$$\begin{aligned}\partial_t&\left[C_v\rho(\theta - \ln\theta - 1) + R(\rho\ln\rho - \rho + 1) + \rho\left(a(\theta) + \left(\frac{Z(\theta)}{2\theta}\right)'\right)q^2 + \frac{1}{2}\rho u^2 + \frac{\tau_1}{4\mu}\rho S_1^2 + \frac{\tau_2}{2\lambda}\rho S_2^2\right]\\ &+\mathrm{div}\left[C_v\rho u(\theta - \ln\theta - 1) + \rho u\left(a(\theta) + \left(\frac{Z(\theta)}{2\theta}\right)'\right)q^2 + \frac{\tau_1}{4\mu}\rho u S_1^2 + \frac{\tau_2}{2\lambda}\rho u S_2^2 + R\rho u\ln\rho - R\rho u\right.\\ &\left.-\frac{q}{\theta} + \frac{1}{2}\rho u|u|^2 + pu + q - Su\right] + \frac{q^2}{\kappa(\theta)\theta^2} + \frac{S_1^2}{2\mu\theta} + \frac{S_2^2}{\lambda\theta} = 0.\end{aligned} \tag{3.7}$$

This equation will later on imply in particular the lower energy estimates of the solutions which are crucial to get the global existence of smooth solutions.

Next, we present some inequalities which are frequently used in the proof of our main results.

Lemma 3.2. *In space dimensions $n = 2, 3$ we have the standard Sobolev imbeddings*
i) $H^2 \hookrightarrow L^\infty$.

ii) $H^1 \hookrightarrow L^p$, for $2 \le p \le 6$.

The following Moser-type inequalities will be used in subsequent sections and can be found as a standard tool for example in [26, 33].

Lemma 3.3. *(i) Let $r, m, n \in \mathbb{N}$, $1 < p \le \infty$, $h \in C^r(\mathbb{R}^m)$, $B := \|h\|_{C^r(\overline{B(0,1)})}$. Then there is a constant $C = C(r, m, n, p) > 0$ such that for all $w = (w_1, \ldots, w_m) \in W^{r,p}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ with $\|w\|_{L^\infty} \le 1$ the inequality*

$$\|\nabla^r h(w)\|_{L^p} \le C\, B\, \|\nabla^r w\|_{L^p} \tag{3.8}$$

holds.

(ii) Let $m \in \mathbb{N}$. Then there is a constant $C = C(m, n) > 0$ such that for all $f, g \in W^{m,2} \cap L^\infty$ and $\alpha \in \mathbb{N}_0^n, |\alpha| \le m$, the following inequalities hold:

$$\|\nabla^{|\alpha|}(fg)\|_2 \le C(\|f\|_{L^\infty}\|\nabla^m g\|_2 + \|\nabla^m f\|_2\|g\|_{L^\infty}), \tag{3.9}$$

$$\|\nabla^{|\alpha|}(fg) - f\nabla^{|\alpha|}g\|_2 \le C(\|\nabla f\|_{L^\infty}\|\nabla^{m-1}g\|_2 + \|\nabla^m f\|_2\|g\|_{L^\infty}). \tag{3.10}$$

4. Uniform estimates and global solutions for small data

Define the energy term by

$$E(t) := \sup_{0\le s\le t}\|(\rho - 1, u, \theta - 1, \sqrt{\tau_0}q, \sqrt{\tau_1}S_1, \sqrt{\tau_2}S_2)\|_{H^3}, \quad E_0 := E(0), \tag{4.1}$$

and the dissipation term by

$$\mathcal{D}(t) := \|(\nabla\rho, \nabla u, \nabla\theta)\|_{H^2}^2 + \|(q, S_1, S_2)\|_{H^3}^2. \tag{4.2}$$

We assume a priori that $E(t) \leq \frac{1}{16\chi^2}$, where χ is the Sobolev embedding constant. Then, we have a priori that

$$\|(\rho-1,\theta-1)\|_{L^\infty} \leq \chi\|(\rho-1,\theta-1)\|_{H^2} \leq \chi\cdot\frac{1}{4\chi} = \frac{1}{4}, \tag{4.3}$$

which implies that $\frac{3}{4} \leq \rho \leq \frac{5}{4}, \frac{3}{4} \leq \theta \leq \frac{5}{4}$. We also assume that the functions $\tilde{\tau}_0(\theta), \kappa(\theta)$ have positive lower and upper bounds if θ has positive lower and upper bounds, that is, for $\frac{3}{4} \leq \theta \leq \frac{5}{4}$,

$$0 < l_1 \leq \tilde{\tau}_0(\theta) \leq l_2 < +\infty, \quad 0 < \kappa_1 \leq \kappa(\theta) \leq \kappa_2 < +\infty. \tag{4.4}$$

Furthermore, there exists a constant δ_1 such that, for $E(t) \leq \delta_1$,

$$a(\theta) + \left(\frac{Z(\theta)}{2\theta}\right)' = \frac{1}{\theta}(1-\frac{1}{2\theta})Z(\theta) + (\frac{1}{\theta}-1)Z'(\theta) \geq C_0\tau_0 \tag{4.5}$$

for some $C_0 > 0$.

Combining (4.3), (4.5), and Remark 3.3, and using Taylor expansion as in [18], we get immediately the following L^2 estimates of solutions.

Lemma 4.1. *There exists a constant $C > 0$ such that, for all $0 \leq t \leq T$,*

$$\int_{\mathbb{R}^3}((\rho-1)^2+(\theta-1)^2+u^2+\tau_0 q^2+\tau_1 S_1^2+\tau_2 S_2^2)\mathrm{d}x + \int_0^t\int_{\mathbb{R}^3}(q^2+S_1^2+S_2^2)\mathrm{d}x\mathrm{d}t \leq CE_0. \tag{4.6}$$

In the following Lemmas, we will alway assume that $E(t) \leq \delta$ for sufficiently small δ. We use the notation $Z(\theta) = \frac{\tau_0(\theta)}{\kappa(\theta)} = \tau_0\frac{\tilde{\tau}_0(\theta)}{\kappa(\theta)} = \tau_0\tilde{Z}(\theta)$ and $a(\theta) = \tau_0\tilde{a}(\theta)$.

Now, we do the higher-order estimates. For $1 \leq |\alpha| \leq 3$, we take ∂_x^α on a slight modification of equations (2.4), and get

$$\begin{cases}\partial_t\partial_x^\alpha\rho + u\cdot\nabla\partial_x^\alpha\rho + \rho\mathrm{div}\partial_x^\alpha u = f_1,\\ \partial_t\partial_x^\alpha u + u\cdot\nabla\partial_x^\alpha u + \frac{p_\rho}{\rho}\nabla\partial_x^\alpha\rho + \frac{p_\theta}{\rho}\nabla\partial_x^\alpha\theta = \frac{1}{\rho}\mathrm{div}\partial_x^\alpha S_1 + \frac{1}{\rho}\nabla\partial_x^\alpha S_2 + f_2,\\ \partial_t\partial_x^\alpha\theta + (u - \frac{2\tilde{a}(\theta)}{\tilde{Z}(\theta)\rho e_\theta}q)\nabla\partial_x^\alpha\theta + \frac{p}{\rho e_\theta}\mathrm{div}\partial_x^\alpha u + \frac{1}{\rho e_\theta}\mathrm{div}\partial_x^\alpha q = \partial_x^\alpha\left(\frac{2\tilde{a}(\theta)}{\tilde{\tau}_0(\theta)\rho e_\theta}q^2 + \frac{1}{2\mu\rho e_\theta}S_1^2 + \frac{1}{\rho e_\theta\lambda}S_2^2\right) + f_3,\\ \tau_0(\partial_t\partial_x^\alpha q + u\cdot\nabla\partial_x^\alpha q) + \frac{1}{\tilde{\tau}_0(\theta)\rho}\partial_x^\alpha q + \frac{\kappa(\theta)}{\tilde{\tau}_0(\theta)\rho}\nabla\partial_x^\alpha\theta = f_4,\\ \tau_1(\partial_t\partial_x^\alpha S_1 + u\cdot\nabla\partial_x^\alpha S_1) + \frac{1}{\rho}\partial_x^\alpha S_1 = \frac{\mu}{\rho}(\nabla\partial_x^\alpha u + \nabla(\partial_x^\alpha u)^T - \frac{2}{3}\mathrm{div}\partial_x^\alpha u I_3) + f_5,\\ \tau_2(\partial_t\partial_x^\alpha S_2 + u\cdot\nabla\partial_x^\alpha S_2) + \frac{1}{\rho}\partial_x^\alpha S_2 = \frac{\lambda}{\rho}\mathrm{div}\partial_x^\alpha u + f_6,\end{cases} \tag{4.7}$$

where $f_i, 1 \le i \le 6$ are commutators given by

$$
\begin{aligned}
f_1 :&= -[\partial_x^\alpha(u\cdot\nabla\rho) - u\cdot\nabla\partial_x^\alpha\rho] - \partial_x^\alpha(\rho\mathrm{div}u) - \rho\mathrm{div}\partial_x^\alpha u,\\
f_2 :&= -[\partial_x^\alpha(u\cdot\nabla u) - u\cdot\nabla\partial_x^\alpha u] - \left[\partial_x^\alpha\left(\frac{p_\rho}{\rho}\nabla\rho\right) - \frac{p_\rho}{\rho}\nabla\partial_x^\alpha\rho\right]\\
&\qquad + \left[\partial_x^\alpha\left(\frac{1}{\rho}(\mathrm{div}S_1 + \nabla S_2)\right) - \frac{1}{\rho}(\mathrm{div}\partial_x^\alpha S_1 + \nabla\partial_x^\alpha S_2)\right]\\
f_3 :&= -\left(\partial_x^\alpha\left(\left(u - \frac{2\tilde{a}(\theta)}{\tilde{Z}(\theta)\rho e_\theta}q\right)\nabla\theta\right) - \left(u - \frac{2\tilde{a}(\theta)}{\tilde{Z}(\theta)\rho e_\theta}q\right)\nabla\partial_x^\alpha\theta\right)\\
&\qquad - \left(\partial_x^\alpha\left(\frac{p}{\rho e_\theta}\mathrm{div}u\right) - \frac{p}{\rho e_\theta}\mathrm{div}\partial_x^\alpha u\right) - \left(\partial_x^\alpha\left(\frac{1}{\rho e_\theta}\mathrm{div}q\right) - \frac{1}{\rho e_\theta}\mathrm{div}\partial_x^\alpha q\right).\\
f_4 :&= -\tau_0(\partial_x^\alpha(u\cdot\nabla q) - u\cdot\nabla\partial_x^\alpha q) - \left(\partial_x^\alpha\left(\frac{1}{\tilde{\tau}_0(\theta)\rho}q\right) - \frac{1}{\tilde{\tau}_0(\theta)\rho}\partial_x^\alpha q\right)\\
&\qquad - \left(\partial_x^\alpha\left(\frac{\kappa(\theta)}{\tilde{\tau}_0(\theta)\rho}\nabla\theta\right) - \frac{\kappa(\theta)}{\tilde{\tau}_0(\theta)\rho}\nabla\partial_x^\alpha\theta\right),\\
f_5 :&= -\tau_1(\partial_x^\alpha(u\cdot\nabla S_1) - u\cdot\nabla\partial_x^\alpha S_1) - \left(\partial_x^\alpha(\frac{1}{\rho}S_1) - \frac{1}{\rho}\partial_x^\alpha S_1\right)\\
&\qquad + \left(\partial_x^\alpha\left(\frac{1}{\rho}\mu(\nabla u + \nabla u^T - \frac{2}{3}\mathrm{div}uI_3)\right) - \frac{\mu}{\rho}(\nabla\partial_x^\alpha u + \nabla(\partial_x^\alpha u)^T - \frac{2}{3}\mathrm{div}\partial_x^\alpha uI_3\right)\\
f_6 :&= -\tau_2\left(\partial_x^\alpha(u\cdot\nabla S_2) - u\cdot\nabla\partial_x^\alpha S_2\right) - \left(\partial_x^\alpha(\frac{1}{\rho}S_2) - \frac{1}{\rho}\partial_x^\alpha S_2\right) - \left(\partial_x^\alpha(\frac{\lambda}{\rho}\mathrm{div}u) - \frac{\lambda}{\rho}\mathrm{div}\partial_x^\alpha u\right)
\end{aligned}
$$

The following lemma gives the estimates on the commutators.

Lemma 4.2. *There is a constant $C > 0$ such that, for all $0 \le t \le T$, we have*

$$\|(f_1, f_2, f_3, f_4, f_5, f_6)\|_{L^2} \le CE^{\frac{1}{2}}(t)D^{\frac{1}{2}}(t). \tag{4.8}$$

Proof. By using the Moser type inequalities and the Sobolev embedding theorem, we get

$$\|\partial_x^\alpha(u\cdot\nabla\rho) - u\cdot\nabla\partial_x^\alpha\rho\|_{L^2} \le C(\|\nabla u\|_{L^\infty}\|\nabla\partial_x^{\alpha-1}\rho\|_{L^2} + \|\nabla\rho\|_{L^\infty}\|\partial_x^\alpha u\|_{L^2}) \le CE^{\frac{1}{2}}(t)D^{\frac{1}{2}}(t).$$

Similarly, we have

$$\|\partial_x^\alpha(\rho\mathrm{div}u) - \rho\mathrm{div}\partial_x^\alpha u\|_{L^2} \le CE^{\frac{1}{2}}(t)D^{\frac{1}{2}}(t).$$

So, we get

$$\|f_1\|_{L^2} \le CE^{\frac{1}{2}}(t)D^{\frac{1}{2}}(t). \tag{4.9}$$

For estimating f_2, we note that

$$\|\partial_x^\alpha(u\cdot\nabla u) - u\cdot\nabla\partial_x^\alpha u\|_{L^2} \le C(\|\nabla u\|_{L^\infty}\|\nabla\partial_x^{\alpha-1}u\|_{L^2} + \|\nabla u\|_{L^\infty}\|\partial_x^\alpha u\|_{L^2}) \le CE^{\frac{1}{2}}(t)D^{\frac{1}{2}}(t),$$

$$\|\partial_x^\alpha\left(\frac{p_\rho}{\rho}\nabla\rho\right) - \frac{p_\rho}{\rho}\nabla\partial_x^\alpha\rho\| \le C\left(\|\nabla(\frac{\theta}{\rho})\|_{L^\infty}\|\nabla\partial_x^{\alpha-1}\rho\|_{L^2} + \|\nabla\rho\|_{L^\infty}\|\partial_x^\alpha(\frac{\rho\theta}{\rho})\|_{L^2}\right) \le CE^{\frac{1}{2}}(t)D^{\frac{1}{2}}(t),$$

and

$$
\begin{aligned}
&\|\partial_x^\alpha(\frac{1}{\rho}(\mathrm{div}S_1 + \nabla S_2)) - \frac{1}{\rho}(\mathrm{div}\partial_x^\alpha S_1 + \nabla\partial_x^\alpha S_2)\|_{L^2}\\
&\le C(\|\nabla\rho\|_{L^\infty}\|(\mathrm{div}\partial_x^{\alpha-1}S_1, \nabla\partial_x^{\alpha-1}S_2)\|_{L^2} + \|(\mathrm{div}S_1, \nabla S_2)\|_{L^\infty}\|\partial_x^\alpha(\frac{1}{\rho})\|_{L^2}) \le CE^{\frac{1}{2}}(t)D^{\frac{1}{2}}(t).
\end{aligned}
$$

So, we derive

$$\|f_2\|_{L^2} \le CE^{\frac{1}{2}}(t)D^{\frac{1}{2}}(t). \tag{4.10}$$

In a similar way, for estimating f_3, by noting that e_θ is a function of $(\theta, \sqrt{\tau_0}q)$, we have

$$\begin{aligned}
&\|\partial_x^\alpha\left(\left(u - \frac{2\tilde{a}(\theta)}{\tilde{Z}(\theta)\rho e_\theta}q\right)\nabla\theta\right) - \left(u - \frac{2\tilde{a}(\theta)}{\tilde{Z}(\theta)\rho e_\theta}q\right)\nabla\partial_x^\alpha\theta\|_{L^2} \\
&\le C(\|(\nabla u, \nabla\rho, \nabla\theta, \sqrt{\tau_0}\nabla q, \nabla q)\|_{L^\infty}\|\nabla\partial_x^{\alpha-1}\theta\|_{L^2} + \|\nabla\theta\|_{L^\infty}\|\partial_x^\alpha(\rho, u, \theta, \sqrt{\tau_0}q)\|_{L^2}) \\
&\le CE^{\frac{1}{2}}(t)D^{\frac{1}{2}}(t),
\end{aligned}$$

$$\begin{aligned}
&\|\partial_x^\alpha(\frac{p}{\rho e_\theta}\mathrm{div}u) - \frac{p}{\rho e_\theta}\mathrm{div}\partial_x^\alpha u\|_{L^2} \\
&\le \|\nabla(\frac{R\theta}{e_\theta})\|_{L^\infty}\|\mathrm{div}\partial_x^{\alpha-1}u\|_{L^2} + \|\mathrm{div}u\|_{L^\infty}\|\partial_x^\alpha(\frac{R\theta}{e_\theta})\|_{L^2} \\
&\le C(\|(\nabla\theta, \nabla\sqrt{\tau_0}q)\|_{L^\infty}\|\mathrm{div}\partial_x^{\alpha-1}u\|_{L^2} + \|\mathrm{div}u\|_{L^\infty}\|\partial_x^\alpha(\theta, \sqrt{\tau_0}q)\|_{L^2}) \le CE^{\frac{1}{2}}(t)D^{\frac{1}{2}}(t)
\end{aligned}$$

and

$$\begin{aligned}
&\|\partial_x^\alpha(\frac{1}{\rho e_\theta}\mathrm{div}q) - \frac{1}{\rho e_\theta}\mathrm{div}\partial_x^\alpha q\| \\
&\le C(\|(\nabla\rho, \nabla\theta, \nabla\sqrt{\tau_0}q)\|_{L^\infty}\|\mathrm{div}\partial_x^{\alpha-1}q\|_{L^2} + \|\mathrm{div}q\|_{L^\infty}\|\partial_x^\alpha(\rho, \theta, \sqrt{\tau_0}q)\|_{L^2} \le CE^{\frac{1}{2}}(t)D^{\frac{1}{2}}(t).
\end{aligned}$$

Therefore, we get

$$\|f_3\|_{L^2} \le CE^{\frac{1}{2}}(t)D^{\frac{1}{2}}(t). \tag{4.11}$$

To estimate f_4, we note

$$\begin{aligned}
&\|\tau_0(\partial_x^\alpha(u\cdot\nabla q) - u\cdot\nabla\partial_x^\alpha q)\|_{L^2} \\
&\le C\tau_0(\|\nabla u\|_{L^\infty}\|\nabla\partial_x^{\alpha-1}q\|_{L^2} + \|\nabla q\|_{L^\infty}\|\partial_x^\alpha u\|_{L^2}) \le CE^{\frac{1}{2}}(t)D^{\frac{1}{2}}(t),
\end{aligned}$$

$$\begin{aligned}
&\|\partial_x^\alpha(\frac{1}{\tilde{\tau}_0(\theta)\rho}q) - \frac{1}{\tilde{\tau}_0(\theta)\rho}\partial_x^\alpha q\| \\
&\le C(\|(\nabla\rho, \nabla\theta)\|_{L^\infty}\|\partial_x^{\alpha-1}q\|_{L^2} + \|q\|_{L^\infty}\|\partial_x^\alpha(\rho, \theta)\|_{L^2}) \le CE^{\frac{1}{2}}(t)D^{\frac{1}{2}}(t)
\end{aligned}$$

and

$$\begin{aligned}
&\|\partial_x^\alpha(\frac{\kappa(\theta)}{\tilde{\tau}_0(\theta)\rho}\nabla\theta) - \frac{\kappa(\theta)}{\tilde{\tau}_0(\theta)\rho}\nabla\partial_x^\alpha\theta\| \\
&\le C(\|(\nabla\rho, \nabla\theta)\|_{L^\infty}\|\nabla\partial_x^{\alpha-1}\theta\|_{L^2} + \|\nabla\theta\|_{L^\infty}\|\partial_x^\alpha(\rho, \theta)\|_{L^2}) \le CE^{\frac{1}{2}}(t)D^{\frac{1}{2}}(t).
\end{aligned}$$

So, we have

$$\|f_4\|_{L^2} \le CE^{\frac{1}{2}}(t)D^{\frac{1}{2}}(t). \tag{4.12}$$

For the estimate of f_5, we have

$$\begin{aligned}
&\|\tau_1(\partial_x^\alpha(u\cdot\nabla S_1) - u\cdot\nabla\partial_x^\alpha S_1)\|_{L^2} \\
&\le C\tau_1(\|\nabla u\|_{L^\infty}\|\nabla\partial_x^{\alpha-1}S_1\|_{L^2} + \|\nabla S_1\|_{L^\infty}\|\partial_x^\alpha u\|_{L^2}) \le CE^{\frac{1}{2}}(t)D^{\frac{1}{2}}(t),
\end{aligned}$$

$$\|\partial_x^\alpha(\frac{1}{\rho})S_1 - \frac{1}{\rho}\partial_x^\alpha S_1\|_{L^2} \le C(\|\nabla\rho\|_{L^\infty}\|\partial_x^{\alpha-1}S_1\|_{L^2} + \|S_1\|_{L^\infty}\|\partial_x^\alpha\rho\|_{L^2}) \le CE^{\frac{1}{2}}(t)D^{\frac{1}{2}}(t)$$

and

$$
\begin{aligned}
&\|\partial_x^\alpha(\frac{\mu}{\rho}(\nabla u+\nabla u^T-\frac{2}{3}\mathrm{div}uI_3))-\frac{\mu}{\rho}(\nabla\partial_x^\alpha u+\nabla(\partial_x^\alpha u)^T-\frac{2}{3}\mathrm{div}\partial_x^\alpha uI_3\|_{L^2}\\
&\le C(\|\nabla\rho\|_{L^\infty}\|\nabla\partial_x^{\alpha-1}u+\nabla(\partial_x^{\alpha-1}u)^T-\frac{2}{3}\mathrm{div}\partial_x^{\alpha-1}uI_3\|_{L^2}+\|\nabla u+\nabla u^T-\frac{2}{3}\mathrm{div}uI_3\|_{L^\infty}\|\partial_x^\alpha\rho\|_{L^2}\\
&\le CE^{\frac12}(t)D^{\frac12}(t).
\end{aligned}
$$

So, we get

$$
\|f_5\|_{L^2}\le CE^{\frac12}(t)D^{\frac12}(t). \tag{4.13}
$$

Similarly, we can estimate f_6 by

$$
\|f_6\|_{L^2}\le CE^{\frac12}(t)D^{\frac12}(t). \tag{4.14}
$$

Combining the above estimates, the proof of Lemma 4.2 is finished. □

The next lemma shows the high-order estimates of the solutions.

Lemma 4.3. *There exists a constant $C>0$ such that, for all $0\le t\le T$, we have*

$$
\begin{aligned}
\|(\nabla\rho,\nabla u,\nabla\theta,\sqrt{\tau_0}\nabla q,\sqrt{\tau_1}\nabla S_1,\sqrt{\tau_2}\nabla S_2)\|_{H^2}^2+\int_0^t\|(\nabla q,\nabla S_1,\nabla S_2)\|_{H^2}^2\mathrm{d}t&\\
\le C(E_0+E^{\frac12}(t)\int_0^t D(t)\mathrm{d}t).&
\end{aligned}\tag{4.15}
$$

Proof. Multiplying the equations (4.7) by $\frac{p_\rho}{\rho^2}\partial_x^\alpha\rho$, $\partial_x^\alpha u$, $\frac{e_\theta}{\theta}\partial_x^\alpha\theta$, $\frac{\tilde\tau_0(\theta)}{\theta\kappa(\theta)}\partial_x^\alpha q$, $\frac{1}{2\mu}\partial_x^\alpha S_1$, $\frac{1}{\lambda}\partial_x^\alpha S_2$, respectively, and integrating with respect to x, we derive

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int\left(\frac{R\theta}{2\rho^2}(\partial_x^\alpha\rho)^2+\frac12(\partial_x^\alpha u)^2+\frac{e_\theta}{2\theta}(\partial_x^\alpha\theta)^2+\frac{\tilde\tau_0(\theta)}{2\theta\kappa(\theta)}(\sqrt{\tau_0}\partial_x^\alpha q)^2+\frac{1}{4\mu}(\sqrt{\tau_1}\partial_x^\alpha S_1)^2+\right.\\
&\left.\frac{1}{2\lambda}(\sqrt{\tau_2}\partial_x^\alpha S_2)^2\right)\mathrm{d}x+\int\left(\frac{1}{\rho\theta\kappa(\theta)}(\partial_x^\alpha q)^2+\frac{1}{2\mu\rho}(\partial_x^\alpha S_1)^2+\frac{1}{\lambda\rho}(\partial_x^\alpha S_2)^2\right)\mathrm{d}x\equiv T+X+L+Q+F,
\end{aligned}
$$

where

$$
T:=\int\left(\partial_t\left(\frac{R\theta}{\rho^2}\right)\cdot\frac12(\partial_x^\alpha\rho)^2+\partial_t\left(\frac{e_\theta}{\theta}\right)\cdot\frac12(\partial_x^\alpha\theta)^2+\partial_t\left(\frac{\tilde\tau_0(\theta)}{\theta\kappa(\theta)}\right)\cdot\frac12\tau_0(\partial_x^\alpha q)^2\right)\mathrm{d}x,
$$

$$
\begin{aligned}
X:=\int&\left(\mathrm{div}\left(\frac{R\theta}{\rho}u\right)\cdot\frac12(\partial_x^\alpha\rho)^2+\mathrm{div}u\cdot\frac12(\partial_x^\alpha u)^2+\mathrm{div}\left(\frac{e_\theta}{\theta}\left(u-\frac{2\tilde a(\theta)}{\tilde Z(\theta)\rho e_\theta}q\right)\right)\cdot\frac12(\partial_x^\alpha\theta)^2\right.\\
&\left.+\mathrm{div}\left(\frac{\tilde\tau_0(\theta)}{\theta\kappa(\theta)}u\right)\cdot\frac12\tau_0(\partial_x^\alpha q)^2+\mathrm{div}\left(\frac{1}{2\mu}u\right)\cdot\frac12\tau_1(\partial_x^\alpha S_1)^2+\mathrm{div}\left(\frac{1}{\lambda}u\right)\cdot\frac12\tau_2(\partial_x^\alpha S_2)^2\right)\mathrm{d}x,
\end{aligned}
$$

$$
\begin{aligned}
L:=\int&\left(\frac{R\theta}{\rho}(\mathrm{div}\partial_x^\alpha u\cdot\partial_x^\alpha\rho+\nabla\partial_x^\alpha\rho\cdot\partial_x^\alpha u)+R(\nabla\partial_x^\alpha\theta\cdot\partial_x^\alpha u+\mathrm{div}\partial_x^\alpha u\cdot\partial_x^\alpha\theta)\right.\\
&+\frac1\rho(\mathrm{div}\partial_x^\alpha S_1\cdot\partial_x^\alpha u+\frac12(\nabla\partial_x^\alpha u+\nabla(\partial_x^\alpha u)^T-\frac23\mathrm{div}\partial_x^\alpha uI_3)\cdot\partial_x^\alpha S_1)\\
&\left.+\frac1\rho(\nabla\partial_x^\alpha S_2\cdot\partial_x^\alpha u+\mathrm{div}\partial_x^\alpha u\cdot\partial_x^\alpha S_2)+\frac{1}{\rho\theta}\left(\mathrm{div}\partial_x^\alpha q\cdot\partial_x^\alpha\theta+\nabla\partial_x^\alpha\theta\cdot\partial_x^\alpha q\right)\right)\mathrm{d}x,
\end{aligned}
$$

$$
Q:=\int\left(\frac{e_\theta}{\theta}\partial_x^\alpha\left(\frac{2\tilde a(\theta)}{\tilde\tau_0(\theta)\rho e_\theta}q^2+\frac{1}{2\mu\rho e_\theta}S_1^2+\frac{1}{\rho e_\theta\lambda}S_2^2\right)\cdot\partial_x^\alpha\theta\right)\mathrm{d}x,
$$

$$F := \int \left(f_1 \frac{p_\rho}{\rho^2}\partial_x^\alpha \rho + f_2 \partial_x^\alpha u + f_3 \frac{e_\theta}{\theta}\partial_x^\alpha \theta + f_4 \frac{\tilde{\tau}_0(\theta)}{\theta\kappa(\theta)}\partial_x^\alpha q + f_5 \frac{1}{2\mu}\partial_x^\alpha S_1 + f_6 \frac{1}{\lambda}\partial_x^\alpha S_2 \right) \mathrm{d}x.$$

We need to estimate each of the above term. Firstly, from the mass and energy equations, we have

$$\begin{aligned}\|\partial_t \rho\|_{L^\infty} &= \|-\mathrm{div}(\rho u)\|_{L^\infty} = \|-u\cdot\nabla\rho - \rho\cdot \mathrm{div}u\|_{L^\infty} \\ &\le \|u\|_{L^\infty}\|\nabla\rho\|_{L^\infty} + \|\rho\|_{L^\infty}\|\mathrm{div}u\|_{L^\infty} \le C(E(t)+E^{\frac12}(t)) \le CE^{\frac12}(t)\end{aligned}$$

and

$$\begin{aligned}&\|\partial_t\theta\|_{L^\infty} \\ &= \|-(u-\frac{2\tilde{a}(\theta)}{\rho e_\theta \tilde{Z}(\theta)})\nabla\theta - \frac{p}{\rho e_\theta}\mathrm{div}u - \frac{1}{\rho e_\theta}\mathrm{div}q + \frac{2}{\kappa\theta\rho e_\theta}q^2 + \frac{1}{2\mu\rho e_\theta}S_1^2 + \frac{1}{\lambda\rho e_\theta}S_2^2\|_{L^\infty} \\ &\le C(\|u\|_{L^\infty}\|\nabla\theta\|_{L^\infty} + \|\nabla\theta\|_{L^\infty} + \|\mathrm{div}u\|_{L^\infty} + \|\mathrm{div}q\|_{L^\infty} + \|q\|_{L^\infty}^2 + \|S_1\|_{L^\infty}^2 + \|S_2^2\|_{L^\infty}^2 \\ &\le C(E^{\frac12}(t)D^{\frac12}(t) + D^{\frac12}(t) + D(t)) \le C(D^{\frac12}(t)+D(t)).\end{aligned}$$

Noting that $e_\theta = C_v + \tilde{a}'(\theta)\tau_0 q^2$ and using the equations (1.9) and (1.13), we obtain

$$\begin{aligned}\|\partial_t(\frac{e_\theta}{\theta})\|_{L^\infty} &= \|\partial_t(\frac{C_v}{\theta} + \frac{\tilde{a}'(\theta)}{\theta}\tau_0 q^2)\|_{L^\infty} \\ &\le C(\|\partial_t\theta\|_{L^\infty} + \|\tau_0 q^2\|_{L^\infty}\|\partial_t\theta\|_{L^\infty} + C\|q\|_{L^\infty}\|\tau_0 q_t\|_{L^\infty} \\ &\le C(D^{\frac12}(t)+D(t)) + CD^{\frac12}(\|\tau_0 u\cdot\nabla q + \frac{\kappa}{\tilde{a}(\theta)}\nabla\theta + \frac{1}{\tilde{a}(\theta)}q\|_{L^\infty} \\ &\le C(D^{\frac12}(t)+D(t)).\end{aligned}$$

Therefore, we get

$$|T| \le C(\|(\partial_t\rho,\partial_t\theta)\|_{L^\infty}\|\nabla\rho\|_{H^2}^2 + \|\partial_t(\frac{e_\theta}{\theta})\|_{L^\infty}\|\nabla\theta\|_{H^2}^2 + \|\partial_t\theta\|_{L^\infty}\tau_0\|\nabla q\|_{H^2}^2) \le CE^{\frac12}(t)D(t). \tag{4.16}$$

To estimate X, we note that

$$\begin{aligned}\int \mathrm{div}(\frac{R\theta}{\rho}u)\cdot\frac12(\partial_x^\alpha\rho)^2\mathrm{d}x &= \int\left(-\frac{R\theta}{\rho^2}u\nabla\rho + \frac{R}{\rho}u\nabla\theta + \frac{R\theta}{\rho}\mathrm{div}u\right)\cdot\frac12(\partial_x^\alpha\rho)^2\mathrm{d}x \\ &\le C\left(\|u\|_{L^\infty}(\|\nabla\rho\|_{L^\infty}+\|\nabla\theta\|_{L^\infty}) + \|\mathrm{div}u\|_{L^\infty}\right)\|\nabla\rho\|_{H^2}^2 \le CE^{\frac12}(t)D(t),\end{aligned}$$

$$\int \mathrm{div}u\cdot\frac12(\partial_x^\alpha u)^2\mathrm{d}x \le C\|\mathrm{div}u\|_{L^\infty}\|\nabla u\|_{H^2}^2 \le CE^{\frac12}(t)D(t),$$

$$\begin{aligned}&\int \mathrm{div}\left(\frac{e_\theta}{\theta}u - \frac{2\tilde{a}(\theta)}{\theta\tilde{Z}(\theta)\rho}\right)\cdot\frac12(\partial_x^\alpha\theta)^2\mathrm{d}x \\ &\le C(\|u\|_{L^\infty} + \|\nabla\theta\|_{L^\infty} + \tau_0\|q\|_{L^\infty}\|\nabla q\|_{L^\infty} + \|\nabla\rho\|_{L^\infty})\|\nabla\theta\|_{H^2}^2 \le CE^{\frac12}(t)D(t),\end{aligned}$$

$$\int \mathrm{div}\left(\frac{\tilde{\tau}_0(\theta)}{\theta\kappa(\theta)}u\right)\cdot\frac12\tau_0(\partial_x^\alpha q)^2\mathrm{d}x \le C(\|\nabla\theta\|_{L^\infty} + \|\mathrm{div}u\|_{L^\infty})\tau_0\|\nabla q\|_{H^2}^2 \le CE^{\frac12}(t)D(t).$$

and

$$\begin{aligned}&\int\left(\mathrm{div}(\frac{u}{2\mu})\cdot\frac12\tau_1(\partial_x^\alpha S_1)^2 + \mathrm{div}(\frac{u}{\lambda})\cdot\frac12\tau_2(\partial_x^\alpha S_2)^2\right)\mathrm{d}x \\ &\le C\|\mathrm{div}u\|_{L^\infty}(\tau_1\|\nabla S_1\|_{H^2}^2 + \tau_2\|\nabla S_2\|_{H^2}^2) \le CE^{\frac12}(t)D(t).\end{aligned}$$

Therefore, we derive that

$$|X| \le CE^{\frac12}(t)D(t). \tag{4.17}$$

Noting that the matrix S_1 is symmetric and has zero trace, we have the following equality

$$\partial_x^\alpha S_1 \cdot \nabla \partial_x^\alpha u = \frac{1}{2}\partial_x^\alpha S_1(\nabla \partial_x^\alpha u + \nabla(\partial_x^\alpha u)^T) = \frac{1}{2}\partial_x^\alpha S_1(\nabla \partial_x^\alpha u + \nabla(\partial_x^\alpha u)^T - \frac{2}{3}\mathrm{div}\partial_x^\alpha u I_3).$$

So, after integrating by parts, we rewrite L as

$$L = -\int \left(\nabla(\frac{R\theta}{\rho})\partial_x^\alpha u \partial_x^\alpha \rho + \nabla(\frac{1}{\rho})\partial_x^\alpha S_1 \partial_x^\alpha u + \nabla(\frac{1}{\rho})\partial_x^\alpha S_2 \partial_x^\alpha u + \nabla(\frac{1}{\rho\theta})\partial_x^\alpha q \partial_x^\alpha \theta\right)\mathrm{d}x$$

which gives

$$\begin{aligned}|L| &\le C\|(\nabla\rho,\nabla\theta)\|_{L^\infty}(\|\nabla u\|_{H^2}^2 + \|\nabla\rho\|_{H^2}^2 + \|\nabla\theta\|_{H^2}^2 + \|\nabla S_1\|_{H^2}^2 + \|\nabla S_2\|_{H^2}^2 + \|\nabla q\|_{H^2}^2)\\ &\le CE^{\frac{1}{2}}(t)D(t). \end{aligned} \tag{4.18}$$

Moreover, it is easy to see that

$$\begin{aligned} Q \le & C(\|q\|_{L^\infty}\|\nabla q\|_{H^2} + \|S_1\|_{L^\infty}\|\nabla S_1\|_{H^2} + \|S_2\|_{L^\infty}\|S_2\|_{H^2}\\ &+ (\|q\|_{L^\infty}^2 + \|S_1\|_{L^\infty}^2 + \|S_2\|_{L^\infty}^2)\|(\nabla\rho,\nabla\theta,\sqrt{\tau_0}\nabla q)\|_{H^2})\|\nabla\theta\|_{H^2} \le CE^{\frac{1}{2}}(t)D(t). \end{aligned} \tag{4.19}$$

Finally, using Lemma 4.2, it holds

$$F \le CE^{\frac{1}{2}}(t)D(t). \tag{4.20}$$

Combining the above estimates, we get the desired result and the proof of Lemma 4.3 is finished. □

In the following Lemmas, we aim to get the dissipative estimates of $\|(\nabla\rho,\nabla u,\nabla\theta)\|_{H^2}^2$.

Lemma 4.4. *There exists a constant $C > 0$ such that, for all $0 \le t \le T$, we have*

$$\begin{aligned} &\frac{2}{5}\mu\|\nabla u\|_{H^2}^2 - \sum_{m=0}^{2}\frac{\mathrm{d}}{\mathrm{d}t}\int \tau_1 \partial_x^\alpha S_1 \cdot \nabla \partial_x^\alpha u \mathrm{d}x\\ &\le \nu(\|\nabla\rho\|_{H^2}^2 + \|\nabla\theta\|_{H^2}^2) + C\|(S_1,S_2)\|_{H^3}^2 + CE^{\frac{1}{2}}(t)D(t), \end{aligned} \tag{4.21}$$

where ν is a small constant to be determined later.

Proof. We replace the notation α by m in equation $(4.7)_5$ and let $0 \le m \le 2$. Then, multiplying the resulting equation by $\nabla\partial_x^m u$ and integrating with respect to x over $\mathbb{R}^3$, we get

$$\begin{aligned} &\int \frac{4\mu}{5}|\nabla\partial_x^m u|^2\mathrm{d}x\\ &\le \int(\tau_1\partial_t\partial_x^m\partial_x^m S_1 \cdot \nabla\partial_x^m u + \tau_1 u \nabla\partial_x^m S_1 \cdot \nabla\partial_x^m u + \frac{1}{\rho}\partial_x^m S_1 \nabla\partial_x^m u - f_5 \cdot \nabla\partial_x^m u)\mathrm{d}x\\ &\le \frac{\mathrm{d}}{\mathrm{d}t}\int \tau_1\partial_x^m S_1 \nabla\partial_x^m u \mathrm{d}x - \int \tau_1 \partial_x^m S_1 \nabla \partial_x^m \partial_t u\\ &\qquad + C\int(|u|^2\|\nabla\partial_x^m S_1|^2 + |\partial_x^m S_1|^2 + |f_5|^2)\mathrm{d}x + \frac{\mu}{5}\|\nabla\partial_x^m u\|_{L^2}^2\\ &\le \frac{\mathrm{d}}{\mathrm{d}t}\int \tau_1\partial_x^m S_1 \nabla\partial_x^m u \mathrm{d}x + \int \tau_1 \partial_x^m \mathrm{div} S_1 \cdot \partial_x^m \partial_t u \mathrm{d}x + C\|S_1\|_{H^3}^2 + CE^{\frac{1}{2}}(t)D(t) + \frac{\mu}{5}\|\nabla\partial_x^m u\|_{L^2}^2, \end{aligned}$$

where we have used the upper bound on ρ (4.3) and the equality

$$\int \nabla\partial_x^m u(\nabla\partial_x^m u + \nabla(\partial_x^m u)^T - \frac{2}{3}\mathrm{div}\partial_x^m u I_3)\mathrm{d}x = \int(|\nabla\partial_x^m u|^2 + \frac{1}{3}|\mathrm{div}\partial_x^m u|^2)\mathrm{d}x.$$

Now, using the momentum equation $(4.7)_2$, we get

$$\begin{aligned}&\int \tau_1 \partial_x^m \mathrm{div} S_1 \partial_x^m \partial_t u \mathrm{d}x \\ &= \int \tau_1 \partial_x^m \mathrm{div} S_1 \left(-u\cdot\nabla\partial_x^m u - \frac{p_\rho}{\rho}\nabla\partial_x^m\rho - \frac{p_\theta}{\rho}\nabla\partial_x^m\theta + \frac{1}{\rho}\mathrm{div}\partial_x^m S_1 + \frac{1}{\rho}\nabla\partial_x^m S_2 + f_2\right)\mathrm{d}x \\ &\le \frac{\mu}{5}\|\nabla\partial_x^m u\|_{L^2}^2 + C\|(S_1,S_2)\|_{H^3}^2 + \nu(\|\nabla\partial_x^m\rho\|_{L^2}^2 + \|\nabla\partial_x^m\theta\|_{L^2}^2) + CE^{\frac12}(t)D(t).\end{aligned}$$

Combining the above estimates and summing m from 0 to 2, we get the desired results. □

Lemma 4.5. *There exists a constant C such that*

$$\frac{2\kappa_1}{5l_2}\|\nabla\theta\|_{H^2}^2 + \sum_{m=0}^{2}\frac{\mathrm{d}}{\mathrm{d}t}\int \tau_0\partial_x^m q\cdot\nabla\partial_x^m\theta \mathrm{d}x \le C(\|\nabla u\|_{H^2}^2 + \|q\|_{H^3}^2) + CE^{\frac12}(t)D(t). \tag{4.22}$$

Proof. Replacing the notation α by m in equation $(4.7)_4$ with $0 \le m \le 2$, multiplying the resulting equation by $\nabla\partial_x^m\theta$, and using Lemma 4.2, one gets

$$\begin{aligned}&\frac{4\kappa_1}{5l_2}\|\nabla\partial_x^m\theta\|_{L^2}^2 \le \int \frac{\kappa(\theta)}{\tilde\tau_0(\theta)\rho}|\nabla\partial_x^m\theta|^2\mathrm{d}x \\ &= -\int\left(\tau_0\partial_t\partial_x^m q\cdot\nabla\partial_x^m\theta + \tau_0 u\cdot\nabla\partial_x^m q\cdot\nabla\partial_x^m\theta + \frac{1}{\tilde\tau_0(\theta)\rho}\partial_x^m q\cdot\nabla\partial_x^m\theta - f_4\nabla\partial_x^m\theta\right)\mathrm{d}x \\ &\le -\frac{\mathrm{d}}{\mathrm{d}t}\int\tau_0\partial_x^m q\cdot\nabla\partial_x^m\theta\mathrm{d}x + \int\tau_0\partial_x^m q\cdot\nabla\partial_x^m\partial_t\theta + \frac{\kappa_1}{5l_2}\|\nabla\partial_x^m\theta\|_{L^2}^2 + C\|q\|_{H^2}^2 + CE^{\frac12}(t)D(t)\end{aligned}$$

Now, using the equation for the temperature $(4.7)_3$, we obtain

$$\begin{aligned}&\int\tau_0\partial_x^m q\cdot\nabla\partial_x^m\partial_t\theta\mathrm{d}x = -\int\tau_0\mathrm{div}\partial_x^m q\cdot\partial_x^m\partial_t\theta\mathrm{d}x \\ &= \int\tau_0\mathrm{div}\partial_x^m q\left(\left(u - \frac{2\tilde a(\theta)}{\tilde Z(\theta)\rho e_\theta}\right)\nabla\partial_x^m\theta + \frac{p}{\rho e_\theta}\mathrm{div}\partial_x^m u + \frac{1}{\rho e_\theta}\mathrm{div}\partial_x^m q\right. \\ &\qquad\qquad \left. -\partial_x^m\left(\frac{2}{\kappa\theta\rho e_\theta}q^2 + \frac{1}{2\mu\rho e_\theta}S_1^2 + \frac{1}{\rho e_\theta\lambda}S_2^2\right) + f_3\right) \\ &\le \frac{\kappa_1}{5l_2}\|\nabla\partial_x^m\theta\|_{L^2}^2 + C(\|q\|_{H^3}^2 + \|\nabla u\|_{H^2}^2) + CE^{\frac12}(t)D(t).\end{aligned}$$

Combining the above estimates and summing m from 0 to 2, we get the desired result (4.22). □

Lemma 4.6. *There exists a constant $C > 0$ such that, for all $0 \le t \le T$, we have*

$$\frac{R}{5}\|\nabla\rho\|_{H^2}^2 + \sum_{m=0}^{2}\frac{\mathrm{d}}{\mathrm{d}t}\int\partial_x^m u\cdot\nabla\partial_x^m\rho \le C(\|\nabla\theta\|_{H^2}^2 + \|\nabla u\|_{H^2}^2 + \|(S_1,S_2)\|_{H^3}^2) + CE^{\frac12}(t)D(t). \tag{4.23}$$

Proof. Replacing the notation α by m in equation $(4.7)_2$ with $0 \le m \le 2$, multiplying the resulting equation by $\nabla\partial_x^m\rho$ and integrating with respect to x over $\mathbb{R}^3$, one gets

$$
\begin{aligned}
&\frac{3R}{5}\|\nabla\partial_x^m\rho\|_{L^2}^2 \le \int \frac{R\theta}{\rho}(\nabla\partial_x^m\rho)^2\mathrm{d}x\\
&= -\int \partial_t\partial_x^m u\cdot\nabla\partial_x^m\rho\mathrm{d}x - \int u\cdot\nabla\partial_x^m u\cdot\nabla\partial_x^m\rho\mathrm{d}x\\
&\qquad\qquad - \int R\nabla\partial_x^m\theta\cdot\nabla\partial_x^m\rho\mathrm{d}x + \int\left(\frac{1}{\rho}(\mathrm{div}\partial_x^m S_1 + \nabla\partial_x^m S_2) + f_2\right)\cdot\nabla\partial_x^m\rho\mathrm{d}x\\
&\le -\frac{\mathrm{d}}{\mathrm{d}t}\int \partial_x^m u\cdot\nabla\partial_x^m\rho\mathrm{d}x - \int \mathrm{div}\partial_x^m u\cdot\partial_x^m\partial_t\rho\mathrm{d}x\\
&\qquad\qquad + C(\|\nabla\theta\|_{H^2}^2 + \|(S_1,S_2)\|_{H^3}^2) + CE^{\frac12}(t)D(t) + \frac{R}{5}\|\nabla\partial_x^m\rho\|_{L^2}^2.
\end{aligned}
$$

Now, using the mass equation $(4.7)_1$, we obtain

$$
\begin{aligned}
\int \partial_x^m\mathrm{div}u\cdot\partial_x^m\partial_t\rho\mathrm{d}x &= -\int \mathrm{div}\partial_x^m u\cdot\partial_x^m\mathrm{div}(\rho u)\mathrm{d}x\\
&\le \frac{R}{5}\|\nabla\partial_x^m\rho\|_{L^2}^2 + C\|\nabla u\|_{H^2}^2 + CE^{\frac12}(t)D(t).
\end{aligned}
$$

Combining the above estimates and summing m from 0 to 2, we get the desired result (4.23). □

Combining Lemmas 4.4-4.6, we have the following

Lemma 4.7. *There exists a constant C such that, for all $0 \le t \le T$, we have*

$$
\int_0^t \|(\nabla\rho,\nabla\theta,\nabla u)\|_{H^2}^2\mathrm{d}t \le C(E_0 + E^{\frac12}(t)\int_0^t D(t)\mathrm{d}t). \tag{4.24}
$$

Proof. Multiplying the inequality (4.23) by $\frac{\kappa_1}{5l_2C}$, and adding the result to the equality (4.22), we get

$$
\begin{aligned}
&\frac{\kappa_1}{5l_2}\|\nabla\theta\|_{H^2}^2 + \frac{R\kappa_1}{25l_2C}\|\nabla\rho\|_{H^2}^2 + \sum_{m=0}^{2}\frac{\mathrm{d}}{\mathrm{d}t}\int\left(\tau_0\partial_x^m q\cdot\nabla\partial_x^m\theta + \frac{\kappa_1}{5l_2C}\partial_x^m u\cdot\nabla\partial_x^m\rho\right)\mathrm{d}x\\
&\le C_1(\|\nabla u\|_{H^2}^2 + \|(q,S_1,S_2)\|_{H^3}^2 + E^{\frac12}(t)D(t)).
\end{aligned}
$$

Multiplying the above result by $\frac{\mu}{5C_1}$, adding the result to the inequality (4.21), choosing ν sufficiently small and integrating with respect to t, we get

$$
\begin{aligned}
&\int_0^t \|(\nabla\rho,\nabla u,\nabla\theta)\|_{H^2}^2\mathrm{d}t\\
&\le C\sum_{m=0}^{2}\int(\tau_1|\partial_x^m S_1|\cdot|\nabla\partial_x^m u| + \tau_0|\partial_x^m q|\cdot|\nabla\partial_x^m\theta| + |\partial_x^m u|\cdot|\nabla\partial_x^m\rho|)\mathrm{d}x\Big|_0^t\\
&\qquad\qquad + \int_0^t \|(q,S_1,S_2)\|_{H^3}^2\mathrm{d}t + CE^{\frac12}(t)\int_0^t D(t)\mathrm{d}t.
\end{aligned}
$$

In view of Lemmas 4.1 and 4.3, we have

$$
\begin{aligned}
&\sum_{m=0}^{2}\int(\tau_1|\partial_x^m S_1|\cdot|\nabla\partial_x^m u| + \tau_0|\partial_x^m q|\cdot|\nabla\partial_x^m\theta| + |\partial_x^m u|\cdot|\nabla\partial_x^m\rho|)\mathrm{d}x\Big|_0^t\\
&\le C(E_0 + \|\sqrt{\tau_1}S_1\|_{H^2}^2 + \|u\|_{H^3}^2 + \|\sqrt{\tau_0}q\|_{H^2}^2 + \|\nabla\theta\|_{H^2}^2 + \|\nabla\rho\|_{H^2}^2 \le C(E_0 + E^{\frac12}(t)\int_0^t D(t)\mathrm{d}t)
\end{aligned}
$$

and

$$\int_0^t \|(q,S_1,S_2)\|_{H^3}^2\mathrm{d}t \le C(E_0+E^{\frac12}(t)\int_0^t D(t)\mathrm{d}t).$$

Combining the above estimates, we arrive at

$$\int_0^t \|(\nabla\rho,\nabla u,\nabla\theta)\|_{H^2}^2\mathrm{d}t \le C(E_0+E^{\frac12}(t)\int_0^t D(t)\mathrm{d}t).$$

□

Proof of Theorem 1.3: Combining Lemmas 4.1, 4.3 and 4.7, we conclude that

$$E(t)+\int_0^t D(t)\mathrm{d}t \le C(E_0+E^{\frac12}(t)\int_0^t D(t)\mathrm{d}t). \tag{4.25}$$

By the usual bootstrap principle and continuation methods, we obtain a global solution satisfying (1.16). Moreover, from the equation (2.4) and inequality (1.16), we have

$$\int_0^t \|\nabla(\rho,u,\theta,\sqrt{\tau_0}q,\sqrt{\tau_1}S_1,\sqrt{\tau_2}S_2)\|_{L^2}^2\mathrm{d}t \le C$$

and

$$\int_0^t \left|\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla(\rho,u,\theta,\sqrt{\tau_0}q,\sqrt{\tau_1}S_1,\sqrt{\tau_2}S_2)\|_{L^2}^2\right|\mathrm{d}t \le C,$$

which implies the decay estimate (1.17) immediately and thus proves Theorem 1.3. □

5. Global relaxation limit

Proof of Theorem 1.4: Assume that $\tau=(\tau_0,\tau_1,\tau_2)$, and let $(\rho^\tau,u^\tau,\theta^\tau,q^\tau,S_1^\tau,S_2^\tau)$ be the global solutions obtained in Theorem 1.3, satisfying

$$\begin{cases}\partial_t\rho^\tau+\mathrm{div}(\rho^\tau u^\tau)=0,\\ \partial_t(\rho^\tau u^\tau)+\mathrm{div}(\rho^\tau u^\tau\otimes u^\tau)+\nabla p^\tau=\mathrm{div}(S_1^\tau+S_2^\tau I_3),\\ \partial_t(\rho^\tau\mathcal{E}^\tau)+\mathrm{div}(\rho^\tau u^\tau\mathcal{E}^\tau+p^\tau u^\tau+q^\tau+S^\tau u^\tau)=0.\\ \tau_0(\theta^\tau)\rho^\tau(\partial_t q^\tau+u^\tau\cdot\nabla q^\tau)+q^\tau+\kappa(\theta^\tau)\nabla\theta^\tau=0,\\ \tau_1\rho^\tau\partial_t(S_1^\tau+u\cdot\nabla S_1^\tau)+S_1^\tau=\mu(\nabla u^\tau+(\nabla u^\tau)^T-\frac23\mathrm{div}u^\tau I_3),\\ \tau_2\rho^\tau\partial_t(S_2^\tau+u\cdot\nabla S_2^\tau)+S_2^\tau=\lambda\mathrm{div}u^\tau\end{cases} \tag{5.1}$$

with

$$\mathcal{E}^\tau=\frac12(u^\tau)^2+\frac{\tau_1}{4\mu}(S_1^\tau)^2+\frac{\tau_2}{2\lambda}(S_2^\tau)^2+e^\tau, \tag{5.2}$$

$$e^\tau=C_v\theta^\tau+a(\theta^\tau)(q^\tau)^2,\quad p^\tau=R\rho^\tau\theta^\tau, \tag{5.3}$$

cf. (1.12), (1.13). By Theorem 1.3 we have

$$\begin{aligned}&\sup_{0\le t<\infty}\|(\rho^\tau-1,u^\tau,\theta^\tau-1,\sqrt{\tau_0}q^\tau,\sqrt{\tau_1}S_1^\tau,\sqrt{\tau_2}S_2^\tau)(t,\cdot)\|_{H^3}^2\\ &+\int_0^\infty\Big(\|(\nabla\rho^\tau,\nabla u^\tau,\nabla\theta^\tau)\|_{H^2}^2+\|(q^\tau,S_1^\tau,S_2^\tau)\|_{H^3}^2\Big)\mathrm{d}t\le CE_0,\end{aligned} \tag{5.4}$$

where C is a constant independent of $\tau=(\tau_0,\tau_1,\tau_2)$.

Thus, there exist $(\rho^0,u^0,\theta^0)\in L^\infty([0,\infty),H^3(\mathbb{R}^3))$ and $(q^0,S_1^0,S_2^0)\in L^2([0,\infty),H^3(\mathbb{R}^3))$ such that

$$(\rho^\tau,u^\tau,\theta^\tau)\longrightarrow(\rho^0,u^0,\theta^0)\quad\text{weak-}*\quad\text{in}\quad L^\infty([0,\infty);H^3(\mathbb{R}^3)),$$

and

$$(q^\tau, S_1^\tau, S_2^\tau) \longrightarrow (q^0, S_1^0, S_2^0) \quad \text{weakly} \quad \text{in} \quad L^2([0,\infty); H^3(\mathbb{R}^3)).$$

Furthermore, it is easy to see that $(\partial_t \rho^\tau, \partial_t u^\tau)$ are bounded in $L^2([0,\infty); H^2(\mathbb{R}^3)$. By classical compactness (ρ^τ, u^τ) are relatively compact in $C([0,T], H^2_{loc}(\mathbb{R}^3))$ for any $T > 0$. As a consequence, as $\tau \to 0$ and up to subsequences,

$$(\rho^\tau, u^\tau) \to (\rho^0, u^0) \quad \text{strongly} \quad \text{in} \quad C([0,T], H^2_{loc}(\mathbb{R}^3)).$$

In contrast, we notice that the boundedness of $\partial_t \theta$ in $L^2([0,\infty); H^2(\mathbb{R}^3))$ is not obtained immediately, so does the strong convergence of θ^τ in $C([0,T], H^2_{loc}(\mathbb{R}^3))$. However, since $\mathcal{E}^\tau$ is a smooth function of $(\theta - 1, \sqrt{\tau_0} q^\tau, \sqrt{\tau_1} S_1^\tau, \sqrt{\tau_2} S_2^\tau, u^\tau)$, $\mathcal{E}^\tau - C_v$ is uniformly bounded in $L^\infty((0,\infty), H^3(\mathbb{R}^3))$. Moreover, from equation $(5.1)_3$, we conclude that $\partial_t \mathcal{E}^\tau$ is uniformly bounded in $L^2((0,\infty); L^2(\mathbb{R}^3)$, the usual compactness argument implies

$$\mathcal{E}^\tau \to \mathcal{E}^0, \quad \text{strongly} \quad \text{in} \quad C([0,T], H^2_{loc}(\mathbb{R}^3)).$$

Thus, by using the formulation (5.2), and the fact that $(\tau_0 (q^\tau)^2, \tau_1 (S_1^\tau)^2, \tau_2 (S_2^\tau)^2)$ are strongly convergent to zero in $L^2([0,T); H^2(\mathbb{R}^3))$ as τ tends to zero, one obtains, for any $T > 0$,

$$\theta^\tau \to \theta^0 \quad \text{strongly} \quad \text{in} \quad L^2([0,T], H^2_{loc}(\mathbb{R}^3)).$$

On the other hand, as $\tau \to 0$, we have

$$\tau_0(\partial_t q^\tau + u \cdot \nabla q) \rightharpoonup 0 \quad \text{in } \mathcal{D}'((0,\infty) \times \mathbb{R}^3),$$

$$\tau_1(\partial_t S_1^\tau + u \cdot \nabla S_1^\tau) \rightharpoonup 0 \quad \text{in } \mathcal{D}'((0,\infty) \times \mathbb{R}^3)$$

and

$$\tau_2(\partial_t S_2^\tau + u \cdot \nabla S_2^\tau) \rightharpoonup 0 \quad \text{in } \mathcal{D}'((0,\infty) \times \mathbb{R}^3).$$

So we may pass to the limit in (5.1)-(5.2) in the sense of distributions. The limits (ρ^0, u^0, θ^0) satisfy the classical compressible Navier-Stokes equations in the distributional sense,

$$\begin{cases} \partial_t \rho^0 + \operatorname{div}(\rho^0 u^0) = 0, \\ \partial_t(\rho^0 u^0) + \operatorname{div}(\rho^0 u^0 \otimes u^0) + \nabla p^0 = \operatorname{div}(S_1^0 + S_2^0 I_3), \\ \partial_t(\rho^0 \mathcal{E}^0) + \operatorname{div}(\rho^0 u^0 \mathcal{E}^0 + p^0 u^0 + q^0 + S^0 u^0) = 0, \end{cases} \tag{5.5}$$

with

$$\mathcal{E}^0 = \frac{1}{2}(u^0)^2 + e^0, \tag{5.6}$$

$$e^0 = C_v \theta^0, \quad p^0 = R \rho^0 \theta^0, \tag{5.7}$$

where q^0, S_1^0, S_2^0 are given by

$$q^0 = -\kappa(\theta^0)\nabla\theta^0, \quad S_1^0 = \mu(\nabla u^0 + \nabla(u^0)^T - \frac{2}{3}\operatorname{div} u^0 I_3), S_2^0 = \lambda \operatorname{div} u^0 \quad \text{a.e.} \quad (0,\infty) \times \Omega. \tag{5.8}$$

Therefore, the proof of Theorem 1.4 is finished. □

6. Blow-up of solutions for large data

Here we now show that there exists large initial data $(\rho_0, u_0, \theta_0, q_0, S_{10}, S_{20})$ such that the local solution $(\rho, u, \theta, q, S_1, S_2)(t,x)$ must blow up in finite time.

Since the system (2.4) is symmetric-hyperbolic, the local solutions possess the finite propagation speed property:

Proposition 6.1. *Let* $(\rho_0, u_0, \theta_0, q_0, S_{10}, S_{20})$ *be given as in Theorem 2.1 and* $(\rho, u, \theta, q, S_1, S_2)$ *be local solutions to* (2.4) *on* $[0, T_0)$. *Let* $M > 0$. *We further assume that the initial data* $(\rho_0 - 1, u_0, \theta_0 - 1, q_0, S_{10}, S_{20})$ *are compactly supported in a ball* $B_0(M)$ *with radius* $M > 0$. *Then, there exists a constant* σ *such that*

$$(\rho(\cdot,t), u(\cdot,t), \theta(\cdot,t), q(\cdot,t), S_1(\cdot,t), S_2(\cdot,t) = (1, 0_{1\times3}, 1, 0_{1\times3}, 0_{3\times3}, 0) := (\bar\rho, \bar u, \bar\theta, \bar q, \bar S_1, \bar S_2)$$

on $D(t) = \{x \in \mathbb{R}^3 \big| |x| \geq M + \sigma t\}, \quad 0 \leq t < T_0$.

Proof of Theorem 1.5: Firstly, the energy equation $(1.1)_3$ implies that G is a constant and

$$G(t) = G(0) > 0. \tag{6.1}$$

In the following, $\int$ denotes $\int_{\mathbb{R}^3}$ for simplicity. By noting that $\operatorname{trace}(S_1) = 0$, and using equations $(1.1)_2$ and (1.11), the constitutive equations (1.13), Remark 3.3 and (6.1), we can derive

$$\begin{aligned}
F'(t) &= \int (\rho u)_t \cdot x \mathrm{d}x - 3\tau_2 \int (\rho S_2)_t \mathrm{d}x \\
&= \int (-\operatorname{div}(\rho u \otimes u) - \nabla p + \operatorname{div} S_1 + \nabla S_2) \cdot x \mathrm{d}x + 3 \int S_2 \mathrm{d}x \\
&= \int \rho |u|^2 \mathrm{d}x + \int 3(p - \bar p) \mathrm{d}x - \int \operatorname{trace}(S_1) \mathrm{d}x \\
&= \int \rho |u|^2 \mathrm{d}x + 3 \int (R\rho\theta - R\bar\rho\bar\theta) \mathrm{d}x \\
&= \int \rho |u|^2 \mathrm{d}x + 3 \int \left(\frac{R}{C_v} (\rho e - \bar\rho \bar e) - \frac{R}{C_v} a(\theta) \rho q^2 \right) \mathrm{d}x \\
&= \int \rho |u|^2 \mathrm{d}x + 3(\gamma - 1) \int (\rho \mathcal{E} - \bar\rho \bar{\mathcal{E}}) \mathrm{d}x - 3(\gamma - 1) \int \frac{1}{2} \left(\rho u^2 + a(\theta) \rho q^2 + \frac{\tau_1}{4\mu} \rho S_1^2 + \frac{\tau_2}{2\lambda} \rho S_2^2 \right) \mathrm{d}x \\
&\geq \frac{5 - 3\gamma}{2} \int \rho |u|^2 \mathrm{d}x - 3(\gamma - 1) \int \left(a(\theta) \rho q^2 + \frac{\tau_1}{4\mu} \rho S_1^2 + \frac{\tau_2}{2\lambda} \rho S_2^2 \right) \mathrm{d}x \\
&\geq \frac{5 - 3\gamma}{2} \int \rho |u|^2 \mathrm{d}x - 3(\gamma - 1)(H_0 + \frac{\max \rho_0}{2} \|u_0\|_{L^2}^2),
\end{aligned}$$

where $\gamma = \frac{R}{C_v} + 1$ and

$$\begin{aligned}
H_0 := \int \Bigg(C_v \rho_0 (\theta_0 - \ln \theta_0 - 1) + R(\rho_0 \ln \rho_0 - \rho_0 + 1) + \rho_0 \left(a(\theta_0) + \frac{1}{2} \left(\frac{Z(\theta_0)}{\theta_0} \right)' \right) q_0^2 \\
+ \frac{\tau_1}{4\mu} \rho_0 S_{10}^2 + \frac{\tau_2}{2\lambda} \rho_0 S_{20}^2 \Bigg) \mathrm{d}x.
\end{aligned} \tag{6.2}$$

On the other hand,

$$\begin{aligned}
F^2(t) &\leq 2 \left(\int \rho u \cdot x \mathrm{d}x \right)^2 + 2\tau_2^2 \left(\int \rho S_2 \mathrm{d}x \right)^2 \\
&\leq 2 \left(\int \rho x^2 \mathrm{d}x \right) \cdot \left(\int \rho |u|^2 \mathrm{d}x \right) + 2\tau_2^2 \int \rho S_2^2 \mathrm{d}x \int \rho \mathrm{d}x \\
&\leq \frac{8\pi \max \rho_0}{3} (M + \sigma t)^5 \int \rho |u|^2 \mathrm{d}x + 4\lambda\tau_2 (H_0 + \frac{\max \rho_0}{2} \|u_0\|_{L^2}^2) \max \rho_0 \cdot \frac{4\pi}{3} (M + \sigma t)^3
\end{aligned}$$

which implies

$$\int \rho |u|^2 \mathrm{d}x \geq \frac{3F^2(t)}{8\pi \max \rho_0 (M + \sigma t)^5} - \frac{2\lambda\tau_2 (H_0 + \frac{\max \rho_0}{2} \|u_0\|_{L^2}^2)}{(M + \sigma t)^2}.$$

Combining the above results, we derive

$$\begin{aligned} F'(t) &\geq \frac{3(5-3\gamma)}{16\pi \max\rho_0 (M+\sigma t)^5} F^2(t) - \left(\frac{(5-3\gamma)\lambda\tau_2}{(M+\sigma t)^2} + 3(\gamma-1)\right)(H_0 + \frac{\max\rho_0}{2}\|u_0\|_{L^2}^2) \\ &\equiv \frac{c_3}{(1+c_2 t)^5} F^2(t) - K(t), \end{aligned} \tag{6.3}$$

where $c_2 := \frac{\sigma}{M}$, $c_3 := \frac{3(5-3\gamma)}{16\pi \max\rho_0 M^5}$. With this Riccati inequality, we can show the blow-up result.

Indeed, assuming a priori that

$$2K(t) \leq \frac{c_3}{(1+c_2 t)^5} F^2(t), \tag{6.4}$$

then we have

$$F'(t) \geq \frac{c_3}{2(1+c_2 t)^5} F^2(t),$$

which gives

$$\frac{1}{F_0} \geq \frac{1}{F_0} - \frac{1}{F(t)} \geq \frac{c_3}{8c_2} - \frac{c_3}{8c_2(1+c_2 t)^4} \tag{6.5}$$

for which the maximal existence time T can not be infinity provided

$$F_0 > \frac{8c_2}{c_3} = \frac{128\pi\sigma \max\rho_0 M^4}{3(5-3\gamma)}. \tag{6.6}$$

Here $F_0 = F(0)$. Moreover, we have

$$\frac{1}{F(t)} \leq \frac{1}{F_0} - \frac{c_3}{8c_2} + \frac{c_3}{8c_2(1+c_2 t)^4},$$

which implies that

$$F(t) \geq \frac{8c_2(1+c_2 t)^4}{c_3}. \tag{6.7}$$

To show that the a priori estimate (6.4) holds, we use the bootstrap method expressed in the following simple lemma, already used in [20].

Lemma 6.2. *Let $f \in C^0([0,\infty),[0,\infty))$ and $0 < a < b$ such that the following holds for any $0 \leq \alpha < \beta < \infty$:*

$$f(0) < a \qquad \text{and} \qquad (\forall\, t \in [\alpha,\beta] : f(t) \leq b \implies \forall\, t \in [\alpha,\beta] : f(t) \leq a.)\,.$$

Then we have

$$\forall t \geq 0 : \ f(t) \leq a.$$

That is, under the a priori assumption (6.4), we need to show that

$$4K(t) \leq \frac{c_3}{(1+c_2 t)^5} F^2(t). \tag{6.8}$$

We need the above equality to hold in particular for $t = 0$, that is,

$$4\left(\frac{(5-3\gamma)\lambda\tau_2}{M^2} + 3(\gamma-1)\right)(H_0 + \frac{\max\rho_0}{2}\|u_0\|_{L^2}^2) \leq c_3 F_0^2. \tag{6.9}$$

Next, using (6.7), one only need to show

$$4K(t)\frac{(1+c_2 t)^5}{c_3} \leq \frac{64c_2^2}{c_3^2}(1+c_2 t)^8 \tag{6.10}$$

for which is sufficient to show

$$\left(\frac{(5-3\gamma)\lambda\tau_2}{M^2}+3(\gamma-1)\right)(H_0+\frac{\max\rho_0}{2}\|u_0\|_{L^2}^2)\leq\frac{16c_2^2}{c_3}. \tag{6.11}$$

Note that (6.6) and (6.11) imply (6.9), we need to find some u_0 such that the assumptions (6.6) and (6.11) hold. Let (cp. [15, 21])

$$\tilde{v}(r)=\begin{cases}L\cos(\frac{\pi}{2}(r-1)), & r\in[0,1],\\ L, & r\in(1,M-1],\\ \frac{L}{2}\cos(\pi(r-M+1))+\frac{L}{2}, & r\in(M-1,M],\\ 0, & r\in(M,+\infty),\end{cases} \tag{6.12}$$

where L is a positive constant to be determined later. $\tilde{v}$ is not in $H^3(\mathbb{R}_+)$, but we can think of $\tilde{v}$ being smoothed around the singular points $r=1,M-1,M$ and put to zero around $r=0$, yielding a function v, with $\|v\|_{L^2}\leq 2\|\tilde{v}\|_{L^2}$. We choose

$$u_0(x):=v(|x|)\frac{x}{|x|}.$$

Assumption (1.22) can easily be satisfied since it is equivalent to requiring

$$\int_{\mathbb{R}^3}\left(\rho_0e_0-\bar{\rho}\bar{e}+\frac{1}{2}u_0^2\right)dx>0,$$

which is satisfied by choosing $\rho_0\theta_0>\bar{\rho}\bar{\theta}=1$. Let $M\geq 5$, then

$$\begin{aligned}\int_{\mathbb{R}^3}x\cdot\rho_0(x)u_0(x)\mathrm{d}x&=\int_{\mathbb{R}^3}\rho_0(x)v(|x|)|x|\mathrm{d}x\\ &\geq\min\rho_0\int_{B_0(M)}v(|x|)|x|\mathrm{d}x\\ &\geq\min\rho_0\int_0^M v(r)r\cdot 4\pi r^2\mathrm{d}r\\ &\geq\min\rho_0\int_2^{M-2}L\cdot 4\pi r^3\mathrm{d}r\geq\frac{\pi\min\rho_0}{32}LM^4.\end{aligned}$$

We choose L (independent of M) sufficiently large such that

$$\begin{aligned}\left|\tau_2\int\rho_0S_{02}\mathrm{d}x\right|&\leq\int_{B_0(M)}\rho_0\mathrm{d}x+\tau_2\int_{B_0(M)}\rho_0S_{02}^2\mathrm{d}x\\ &\leq\frac{4\pi}{3}M^3\max\rho_0+2\lambda\max\rho_0H_0\leq\frac{\pi\min\rho_0}{64}LM^4\end{aligned}$$

and

$$\frac{\pi\min\rho_0}{64}L\geq\frac{128\sigma\pi\max\rho_0}{3(5-3\gamma)}.$$

So, (6.6) holds. Now, after having chosen L large enough, fix L. On the other hand, since $\|u_0\|_{L^2}^2\leq 4L^2\frac{4\pi}{3}M^3$, Then we choose M sufficiently large and $\gamma-1$ sufficiently small such that (6.11) holds. This finishes the proof of Theorem 1.5. □

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References

[1] S. Boyaval, Viscoelastic flows of Maxwell fluids with conservation laws, *ESAIM: M2AN* **55**(2021), 807-831.

[2] C. Cattaneo, Sulla conduzione del calore, *Atti Sem. Mat. Fis. Univ. Modena* **3** (1948), 83-101.

[3] D. Chakraborty and J.E. Sader, Constitutive models for linear compressible viscoelastic flows of simple liquids at nanometer length scales, *Physics of Fluids* **27** (2015), 052002-1-052002-13.

[4] P.J. Chen and M.E. Gurtin, On second sound in materials with memory, *Z. Ang. Math. Phys.* **21** (1970), 232-241.

[5] Y. Cho and B.J. Jin, Blow-up of viscous heat-conducting compressible flows, *J. Math. Anal. Appl.* **320** (2) (2006), 819-826.

[6] C.I. Christov, On frame indifferent formulation of the Maxwell-Cattaneo model of finite-speed heat conduction, *Mech. Res. Comm.* **36** (2009), 481-486.

[7] B.D. Coleman, M. Fabrizio and D.R. Owen, On the thermodynamics of second sound in dielectric crystals, *Arch. Rational Mech. Anal.* **80** (1982), 135-158.

[8] B.D. Coleman, W.J. Hrusa and D.R. Owen, Stability of Equilibrium for a Nonlinear Hyperbolic System Describing Heat Propagation by Second Sound in Solids, *Arch. Rational Mech. Anal.* **94** (1986), 267-289.

[9] E. Feireisl, A. Novotny and H. Petzeltová, On the existence of globally defined weak solutions to the Navier-Stokes equations, *J. Math. Fluid Mech.* **3** (2001), 358-392.

[10] H. Freistühler, A Galilei invariant version of Yong's model. arXiv 2012.09059 (2020).

[11] H. Freistühler, Time-asymptotic stability for first-order symmetric hyperbolic systems of balance laws in dissipative compressible fluid dynamics, *Quart. Appl. Math.* **80** (2022), 597-606.

[12] C. Guillopé and J.C. Saut, Existence resuls for the flow of viscoelastic fluids with a differential constitutive law, *Nonlinear Analysis, TMA* **15** (1990), 849-869.

[13] D. Hoff, Global existence for 1D, compressible, isentropic Navier-Stokes equations with large initial data, *Trans. Amer. Math. Soc.* **303** (1) (1987), 169-181.

[14] D. Hoff, Global solutions of the Navier-Stokes equations for multidimensional compressible flow with discontinuous initial data, *J. Differential Equations* **120** (1) (1995), 215-254.

[15] Y. Hu and R. Racke, Formation of singularities in one-dimensional thermoelasticity with second sound, *Quart. Appl. Math.* **72** (2014), 311-321.

[16] Y. Hu and R. Racke, Compressible Navier-Stokes equations with hyperbolic heat conduction, *J. Hyper. Diff. Equ.* **13** (2016), 233-247.

[17] Y. Hu and R. Racke, Compressible Navier-Stokes equations with revised Maxwell's law, *J. Math.Fluid Mech.* **19** (2017), 77-90.

[18] Y. Hu and R. Racke, Hyperbolic compressible Navier-Stokes equations, *J. Differential Equations* **269** (2020), 3196-3220.

[19] Y. Hu and R. Racke, Global existence versus blow-up for multidimensional hyperbolized compressible Navier-Stokes equations, *SIAM J. Math. Anal.* **55** (2023), 4788-4815.

[20] Y. Hu and R. Racke, Blow-up of solutions for relaxed compressible Navier-Stokes equations, *J. Hyper. Diff. Equ.* **21** (2024), 129-141.

[21] Y. Hu, R. Racke and N. Wang, Formation of singularities for one-dimensional relaxed compressible Navier-Stokes equations, *J. Differential Equations* **327** (2022), 145-165.

[22] S. Jiang and P. Zhang, Global spherically symmetry solutions of the compressible isentropic Navier-Stokes equations, *Comm. Math. Phys.* **215** (2001), 559-581.

[23] S. Jiang and P. Zhang, Axisymmetric solutions of the 3-D Navier-Stokes equations for compressible isentropic fluids, *J. Math. Pures. Appl.* **82** (2003), 949-973.

[24] S. Kawashima, *Systems of a hyperbolic-parabolic composite type, with applications to the equations of magnetohydrodynamics,* Thesis, Kyoto University (1983).

[25] P.L. Lions, *Mathematical Topics in Fluid Mechanics*, Vol.II, Compressible Models. Clarendon Press, Oxford (1998).

[26] A. Majda, *Compressible fluid flow and systems of conservation laws in several space variables.* Appl. Math. Sci. **53**, Springer, New York (1984).

[27] A. Matsumura and T. Nishida, The initial value problem for the equations of motion of viscous and heat-conductive gases, *J. Math. Kyoto Univ.* **20** (1) (1980), 67-104.

[28] I. Müller, Zum Paradoxon der Wärmeleitungstheorie, Zeitschrift für Physik, **198** (1967), 329-344.

[29] J.C. Maxwell, On the dynamics theory of gases, *Phil. Trans. R. Soc. Lond.* **157** (1867), 49-88.

[30] J. Nash, Le problème de Cauchy pour les équations différentielles d'un fluide général, *Bull. Soc. Math. France* **90** (1962), 487-497.

[31] M. Pelton, D. Chakraborty, E. Malachosky, P. Guyot-Sionnest and J. E. Sader, Viscoelastic flows in simple liquids generated by vibrating nanostructures, *Phys. Rev. Lett.*, **111** (2013), 244502.

[32] Y.-J. Peng and L. Zhao, Global convergence to compressible full Navier–Stokes equations by approximation with Oldroyd-type constitutive laws, *J. Math. Fluid Mech.* **24** (2022), 24:29.
[33] R. Racke, *Lectures on Nonlinear Evolution Equations. Initial Value Problems,* 2nd edition, Birkhäuser, Basel (2015).
[34] T. Ruggeri, Symmetric-hyperbolic system of conservative equations for a viscous heat conducting fluid. *Acta Mech.* **47** (1983), 167-183.
[35] J. Serrin, On the uniqueness of compressible fluid motion, *Arch. Rational Mech. Anal.* **3** (1959), 271-288.
[36] T.C. Sideris, Formation of singularities in three-dimensional compressible fluids, *Commun. Math. Phys.* **101** (1985), 475-485.
[37] M.A. Tarabek, On the existence of smooth solutions in one-dimensional nonlinear thermoelasticity with second sound, *Quart. Appl. Math.* **50** (1992), 727-742.
[38] Z.P. Xin, Blowup of smooth solutions to the compressible Navier-Stokes equation with compact density, *Comm. Pure. Appl. Math.* **51** (1998), 229-240.
[39] W.A. Yong, Newtonian limit of Maxwell fluid flows, *Arch. Rational Mech. Anal.* **214** (2014), 913-922.