Combining the SOS and SONC cones - A Hilbert's 1888 Theorem analogue and further separation results

Moritz Schick *

* *University of Konstanz, Konstanz, Germany (e-mail: moritz.schick@uni-konstanz.de).*

Abstract: Studying convex cones inside the cone of positive semidefinite (PSD) polynomials is an important field of research in real algebraic geometry and polynomial optimization. In this work, we combine two such well established cones, which are sums of squares (SOS) and sums of nonnegative circuit polynomials (SONC) and consider PSD polynomials, that decompose into an SOS and a SONC part. We call the resulting set the SOS+SONC cone. For this newly established cone, we prove two separation results. The first one is an analogue to Hilbert's 1888 Theorem for the SOS+SONC cone. The second one shows that whenever the SOS and SONC cones are proper subsets of the PSD cone, they are also proper subsets of the SOS+SONC cone.

Keywords: sums of squares, sums of nonnegative circuit polynomials, Hilbert's 1888 Theorem, polynomial optimization

1. INTRODUCTION

Minimizing a given real, multivariate polynomial $f \in \mathbb{R}[\mathbf{x}]$ is the central challenge of polynomial optimization. The importance of this topic can be seen in the variety of applications in different fields such as optimal control, mathematical finance and real-time decision making. It is well known that polynomial optimization can equivalently be viewed as the problem of deciding nonnegativity of real polynomials. This equivalence is of central meaning in real algebraic geometry since convex geometric tools can be used to obtain a deeper understanding of the set of nonnegative polynomials.

The relevance of a theoretical study of both problems is also stressed by the fact that both polynomial optimization and deciding nonnegativity of real polynomials are in general NP-hard even for low dimensional cases. Hence, one is often interested in solving easier problems instead, often involving trade offs between feasibility and preciseness of solutions. In this work we follow the real algebraic geometric approach of taking suitable inner approximations of the set of nonnegative polynomials.

A first inner approximation of the cone of nonnegative polynomials are *sums of squares (SOS)*, which have a long history in Mathematics and go back to Hilbert's seminal work in Hilbert (1888). The SOS approach has proven to be a powerful tool for solving a vast number of optimization problems, see see e.g. Lasserre (2009) for more details. However, it has its limitations especially in high degree and high number of variables cases.

A second approximation which has gained a lot of interest in recent years are *sums of nonnegative circuit (SONC) polynomials*, which were first introduced by Iliman and de Wolff (2016). The sparse structure of this class of polynomials allows to solve large problems, where the SOS approach has its difficulties. However, since the SONC approach is relatively new, it is only fully developed for special classes of polynomials, having e.g. simplex Newton polytopes. Indeed, there are different types of conic programming using SONC for polynomial optimization, see e.g. Wang and Magron (2020) and Dressler et al. (2020).

Dressler (2018) pointed out that if it would be possible to combine the two approaches and use the best of both worlds, one would get a new approximation which is at least as good as the single approaches themselves.

In terms of polynomial optimization, let $f \in \mathbb{R}[\mathbf{x}]$ and consider the global polynomial optimization problem $f^* = \inf_{x\in\mathbb{R}^n} f(x) = \sup\{\lambda \in \mathbb{R} : f - \lambda \geq 0 \text{ on } \mathbb{R}^n\}$. Let SOS and SONC denote the sets of SOS and SONC polynomials, respectively. Then, lower bounds on f^* can be achieved via $f_{\text{SOS}} := \sup\{\lambda \in \mathbb{R} : f - \lambda \in \text{SOS}\}$ and $f_{\text{SONC}} := \sup\{\lambda \in \mathbb{R} : f - \lambda \in \text{SONC}\}$. Taking the Minkowski sum $\text{SOS} + \text{SONC}$ leads to a third lower bound $f_{\text{SOS+SONC}} := f_{\text{SOS}\,+\,\text{SONC}} := \sup\{\lambda \in \mathbb{R} : f - \lambda \in \text{SOS} + \text{SONC}\}$ which satisfies $f_{\text{SOS}}, f_{\text{SONC}} \leq f_{\text{SOS+SONC}} \leq f^*$.

In this work, we fully characterize the numbers of variables and degrees of polynomials for which the above's inequalities are strict. Therefore, we formally introduce the cone of sums of squares and nonnegative polynomials (SOS+SONC) and present explicit examples of polynomials separating this cone from the SOS and SONC cones as well as the cone of positive semidefinite polynomials.

2. PRELIMINARIES

2.1 Notations

Let $\mathbb{N} := \{1, 2, 3, \ldots\}$ and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ be the sets of positive and nonnegative integers, respectively and $[m] := \{1, \ldots, m\}$ $(m \in \mathbb{N})$. For $n \in \mathbb{N}$ let $\mathbb{R}[\mathbf{x}] := \mathbb{R}[\mathrm{x}_1, \ldots, \mathrm{x}_n]$

be the polynomial ring over $\mathbb{R}$ in n variables. The integer $n \in \mathbb{N}$ will be fixed throughout this work.

A polynomial $f \in \mathbb{R}[\mathbf{x}]$ with all monomials having the same degree $k \in \mathbb{N}$ is called *homogeneous* or a *form*. For $k \in \mathbb{N}$ we denote by $H_{n,k}$ the finite dimensional vector space of n-variate, homogeneous polynomials of degree k. For $f \in \mathbb{R}[\mathbf{x}]$, we write $\operatorname{supp}(f)$ for the *support* and $\operatorname{New}(f)$ for the *Newton polytope* of f, i.e. $\operatorname{New}(f) = \operatorname{conv}(\operatorname{supp}(f))$, where $\operatorname{conv}(S)$ is the *convex hull* of a set S. For an arbitrary polytope $\Delta \subseteq \mathbb{R}^n$, we denote its set of *vertices* by $V(\Delta)$. If $\Delta = \operatorname{New}(f)$ is the Newton polytope of a polynomial f, we also write $V(f) := V(\operatorname{New}(f))$. For $\alpha \in \mathbb{N}_0^n$ and $f \in \mathbb{R}[\mathbf{x}]$, we denote by f_α the coefficient of f corresponding to x^α, i.e. $f = \sum_{\alpha \in \mathbb{N}_0^n} f_\alpha \mathrm{x}^\alpha$ and $f_\alpha = 0$ if $\alpha \notin \operatorname{supp}(f)$.

2.2 Young's Inequality

The following Theorem is essential for proofs in Section 3.

Young's Inequality Let $1 < p, q < \infty$ be s.t. $1/p + 1/q = 1$. Further, let $a, b \in \mathbb{R}$ be arbitrary. Then $ab \leq \frac{|a|^p}{p} + \frac{|b|^q}{q}$. Further, for $a, b \geq 0$ equality holds if and only if $a^p = b^q$.

2.3 Positive Semidefinite (PSD) Polynomials

A polynomial $f \in \mathbb{R}[\mathbf{x}]$ is *positive semidefinite (PSD)* if $f(x) \geq 0$ for all $x \in \mathbb{R}^n$. Clearly, a PSD polynomial must have even degree. Therefore, an even integer $2d \in 2\mathbb{N}$ will be fixed throughout this work.

We denote by

$$P_{n,2d} := \{f \in H_{n,2d} : f \geq 0 \text{ on } \mathbb{R}^n\}$$

the set of PSD forms of degree $2d$. It is well known that $P_{n,2d}$ is a closed, convex cone in the vector space $H_{n,2d}$.

2.4 Sum of Squares (SOS) Polynomials

A polynomial $f \in \mathbb{R}[\mathbf{x}]$ is a *sum of squares (SOS)* if it admits a decomposition of the form $f = \sum_{i=1}^s f_i^2$ such that $s \in \mathbb{N}$, $f_i \in \mathbb{R}[\mathbf{x}]$ $(i \in [s])$. The set of SOS forms of degree $2d$ is denoted by

$$\Sigma_{n,2d} := \left\{ f = \sum_{i=1}^s f_i^2 : f_i \in \mathbb{R}[\mathbf{x}]_{\leq d}, s \in \mathbb{N} \right\}.$$

Similar as the PSD cone, $\Sigma_{n,2d}$ forms a closed, convex cone inside $P_{n,2d}$.

The following theorem characterizes precisely the cases of $(n, 2d)$ for which the PSD and SOS cones coincide. It goes back to Hilbert's work Hilbert (1888).

Hilbert 1888 It holds $\Sigma_{n,2d} = P_{n,2d}$ if and only if $n = 2$ or $2d = 2$ or $(n, 2d) = (3, 4)$.

2.5 Sums of Nonnegative Circuit (SONC) Polynomials

A polynomial $f = \sum_{i=1}^m c_i \underline{\mathrm{x}}^{\alpha(i)} + b\underline{\mathrm{x}}^\beta \in \mathbb{R}[\mathbf{x}]$ where $m \in \mathbb{N}$, $\alpha(1), \dots, \alpha(m)$, $\beta \in \mathbb{N}_0^n$ and $c_1, \dots, c_m, b \in \mathbb{R}$ is a *circuit polynomial* if it satisfies the following conditions:

(C1) The lattice points $\alpha(1), \dots, \alpha(m)$ are even, i.e. $\alpha(1), \dots, \alpha(m) \in (2\mathbb{N}_0)^n$ and affinely independent.
(C2) The coefficients c_i corresponding to the $\alpha(i)$ are positive, i.e. $c_i > 0$ for $i = 1, \dots, m$.
(C3) The exponent β lies in the interior of the Newton polytope of f.

The monomials $c_i \mathrm{x}^{\alpha(i)}$ $(i = 1, \dots, m)$ are called *vertex monomials*. In addition, if $b \neq 0$ the monomial $b\mathrm{x}^\beta$ is called *interior monomial*. The set of circuit polynomials supported on $A \subseteq \mathbb{N}_0^n$ is denoted by $\operatorname{Circ}_A \subseteq \mathbb{R}[\underline{\mathrm{x}}]$.

A polynomial $f \in \mathbb{R}[\mathbf{x}]$ is a *sum of nonnegative circuit polynomials (SONC)* if it admits a decomposition of the form $f = \sum_{i=1}^s f_i$ where $s \in \mathbb{N}$ and every f_i is a nonnegative circuit polynomial. We denote by

$$C_{n,2d} := \left\{ f = \sum_{i=1}^s f_i : f_i \in \operatorname{Circ}_{A_i} \cap P_{n,2d}, A_i \subseteq \mathbb{N}_0^n \right\}$$

the set of SONC polynomials of degree at most $2d$. Again, $C_{n,2d}$ is a closed, convex cone inside $P_{n,2d}$.

The following Theorem shows that $C_{n,2d}$ is an inner approximation of $P_{n,2d}$, which is independent of $\Sigma_{n,2d}$. It is a combination of results in (Iliman and de Wolff, 2016, Prop. 7.2) and (Dressler, 2018, Thm. 3.1.2)

Theorem The SONC cone $C_{n,2d}$ satisfies:

(1.) $C_{n,2d} \subseteq \Sigma_{n,2d}$ if and only if $n = 2$ or $2d = 2$ or $(n, 2d) = (3, 4)$.
(2.) $C_{2,2} = \Sigma_{2,2}$ and $\Sigma_{n,2} \not\subseteq C_{n,2}$ for all $n \geq 3$.
(3.) $\Sigma_{n,2d} \not\subseteq C_{n,2d}$ for all $(n, 2d)$ with $2d \geq 4$.

2.6 The combined cone SOS+SONC of sums of squares and nonnegative circuit polynomials

A polynomial $f \in \mathbb{R}[\mathbf{x}]$ is said to be a *sum of squares and nonnegative circuit polynomials (SOS+SONC)* if it has a decomposition of the form $f = g + h$ for some $f \in \Sigma_{n,2d}, g \in C_{n,2d}$. Further, we denote by

$$(\Sigma + C)_{n,2d} := \Sigma_{n,2d} + C_{n,2d}$$

the set of SOS+SONC forms in n variables of degree $2d$.

3. SEPARATING THE PSD CONE FROM THE SOS+SONC CONE - A SOS+SONC ANALOGUE TO HILBERT'S 1888 THEOREM

As in *Hilbert 1888* for the SOS case and the theorem of Section 2.5 for the SONC case, it is of interest to find separation results, which classify whenever a given inner approximation of the PSD cone is proper or not. Therefore, we show in Theorem 1 an analogue to *Hilbert 1888* for the SOS+SONC case. The following statement was already proven in a non constructive way in (Averkov, 2019, Corollary 2.17). As a contribution we present an alternative proof by constructing appropriate polynomials in the two basic cases $((n, 2d) \in \{(3, 6), (4, 4)\})$ and scaling them to higher dimensional and number of variables cases.

Theorem 1. It holds $(\Sigma + C)_{n,2d} = P_{n,2d}$ if and only if $n = 2$ or $2d = 2$ or $(n, 2d) = (3, 4)$.

As a first step, we show that it suffices to consider the two elementary cases of ternary sextics and quarternary quartics, i.e. $(n, 2d) \in \{(3, 6), (4, 4)\}$.

Lemma 2. If $(\Sigma + C)_{3,6} \subsetneq P_{3,6}$ and $(\Sigma + C)_{4,4} \subsetneq P_{4,4}$ then $(\Sigma_{n,k} + C)_{n,k} \subsetneq P_{n,k}$ for all $n \geq 3, k \geq 4$ and $(n,k) \neq (3,4)$ (k even).

Proof. We show equivalently: $P_{3,6} \backslash (\Sigma + C)_{3,6} \neq \emptyset$ and $P_{4,4} \backslash (\Sigma + C)_{4,4} \neq \emptyset$ imply $P_{n,k} \backslash (\Sigma + C)_{n,k} \neq \emptyset$ for all $n \geq 3, k \geq 4$ and $(n,k) \neq (3,4)$ where k is even.

Claim 1: $f \in P_{n,k} \backslash (\Sigma + C)_{n,k}$ implies for all $m \in \mathbb{N}$, $f \in P_{n+m,k} \backslash (\Sigma + C)_{n+m,k}$.

The case $m = 1$ can be seen easily, since SOS and SONC polynomials in $H_{n+1,k}$ stay SOS and SONC, respectively, after plugging in $\mathrm{x}_{n+1} = 0$. The general case follows inductively.

Claim 2: $f \in P_{n,k} \backslash (\Sigma + C)_{n,k}$ implies for all $\ell \in \mathbb{N}$ $\mathrm{x}_1^{2\ell} f \in P_{n,k+2\ell} \backslash (\Sigma + C)_{n,k+2\ell}$.

Consider $\ell = 1$ and let $f \in P_{n,k} \backslash (\Sigma + C)_{n,k}$ be arbitrary. Assume that $\mathrm{x}_1^2 f \in (\Sigma + C)_{n,k+2}$, i.e. there is a decomposition $\mathrm{x}_1^2 f = f_{\mathrm{SOS}} + f_{\mathrm{SONC}}$ with f_{SOS} and f_{SONC} being SOS and SONC polynomials in $H_{n,k+2}$, respectively. Since the left hand side vanishes at $\mathrm{x}_1 = 0$, the right hand side must vanish at $\mathrm{x}_1 = 0$ as well. Since $f_{\mathrm{SOS}}, f_{\mathrm{SONC}}$ are PSD, we obtain $f_{\mathrm{SOS}}(0, \mathrm{x}_2, \ldots, \mathrm{x}_n) = 0 = f_{\mathrm{SONC}}(0, \mathrm{x}_2, \ldots, \mathrm{x}_n)$. Hence $\mathrm{x}_1 \mid f_{\mathrm{SOS}}, f_{\mathrm{SONC}}$, i.e. $f_{\mathrm{SOS}} = \mathrm{x}_1 \cdot f_1, f_{\mathrm{SONC}} = \mathrm{x}_1 \cdot f_2$ for some $f_1, f_2 \in H_{n,k+1}$.

Write $f_{\mathrm{SOS}} = \sum_{i=1}^{s} g_i^2,\ f_{\mathrm{SONC}} = \sum_{j=1}^{t} h_j, s, t \in \mathbb{N}$ s.t. $g_i \in H_{n,k/2+1}$ and the h_j are nonnegative circuit polynomials. Similary as above, PSDness of g_i^2 and h_j yields $\mathrm{x}_1 \mid g_i^2, h_j$. For the SOS part, we immediately obtain $\mathrm{x}_1 \mid g_i$, i.e. $\mathrm{x}_1^2 \mid f_{\mathrm{SOS}}$. Remains to show $\mathrm{x}_1^2 \mid h_j$ for all j. However, this follows since all monomials of h_j must be divisible by x_1, all vertex monomials of x_1 are even lattice points and the only interior monomial is a convex combination of the vertex monomials. Hence we have $\mathrm{x}_1^2 \mid f_{\mathrm{SOS}}, f_{\mathrm{SONC}}$, which yields that $f = f_{\mathrm{SOS}}/\mathrm{x}_1^2 + f_{\mathrm{SONC}}/\mathrm{x}_1^2$ would be a SOS+SONC decomposition in $H_{n,k}$, a contradiction. This shows Claim 2 for $\ell = 1$. The general case follows inductively.

Combining Claim 1 and Claim 2 shows the Lemma.

Next, we cover the two elementary cases where $(n, 2d) \in \{(3,6), (4,4)\}$. Therefore, we show that the two Robinson forms from Robinson (1969) are indeed PSD but not SOS+SONC.

Lemma 3. It holds

$$R_1(\mathrm{x},\mathrm{y},\mathrm{z}) = \mathrm{x}^6 + \mathrm{y}^6 + \mathrm{z}^6 - \big(\mathrm{x}^4\mathrm{y}^2 + \mathrm{x}^4\mathrm{z}^2 + \mathrm{y}^4\mathrm{x}^2 + \mathrm{y}^4\mathrm{z}^2 + \mathrm{z}^4\mathrm{x}^2 + \mathrm{z}^4\mathrm{y}^2\big) + 3\mathrm{x}^2\mathrm{y}^2\mathrm{z}^2 \in P_{3,6} \backslash (\Sigma + C)_{3,6}$$

and hence $P_{3,6} \neq (\Sigma + C)_{3,6}$.

Proof. Step 1: Assume that $R_1 \in (\Sigma + C)_{3,6}$ was SOS+SONC. Choose an SOS polynomial $f_{\mathrm{SOS}} \in \Sigma_{3,6}$ such that $R_1 - f_{\mathrm{SOS}} \in C_{3,6}$ is a SONC polynomial. Without loss of generality we can assume that $R_1 - f_{\mathrm{SOS}}$ decomposes into nonnegative circuit polynomials, which are not SOS. Since $R_1 = f_{\mathrm{SOS}} + (R_1 - f_{\mathrm{SOS}})$ is a decomposition into PSD polynomials, $\mathrm{New}(f_{\mathrm{SOS}})$, $\mathrm{New}(R_1 - f_{\mathrm{SOS}}) \subseteq \mathrm{New}(R_1)$ must hold (cf. (Reznick, 1978, Theorem 1)).

Step 2: By *Hilbert 1888*, we know that every PSD bivariate form is SOS, i.e. $P_{2,2d} = \Sigma_{2,2d}$ for all $d \in \mathbb{N}$. Now consider e.g. the monomial $m = \mathrm{x}^4\mathrm{y}^2$. Assume that the SONC part $R_1 - f_{\mathrm{SOS}}$ contains a nonnegative circuit polynomial h having m as interior monomial, i.e. $h_{(4,2,0)'} \lneq 0$. But then, since m is in the interior of $\mathrm{New}(h)$, the circuit polynomial h must clearly be bivariate, i.e. $h \in C_{2,6} \subseteq P_{2,6} = \Sigma_{2,6}$. This contradicts our assumption that no nonnegative circuit polynomial in $R_1 - f_{\mathrm{SOS}}$ is also SOS. Hence, m cannot be an interior monomial of any nonnegative circuit polynomials in the decomposition of $R_1 - f_{\mathrm{SOS}}$, which shows that $(R_1 - f_{\mathrm{SOS}})_{(4,2,0)'} \geq 0$ and equivalently $(f_{\mathrm{SOS}})_{(4,2,0)'} \leq -1$. Analogous argumentation shows

$$\left.\begin{array}{l} (f_{\mathrm{SOS}})_{(4,2,0)'},\ (f_{\mathrm{SOS}})_{(4,0,2)'},\ (f_{\mathrm{SOS}})_{(2,4,0)'}, \\ (f_{\mathrm{SOS}})_{(2,0,4)'},\ (f_{\mathrm{SOS}})_{(0,4,2)'},\ (f_{\mathrm{SOS}})_{(0,2,4)'} \end{array}\right\} \leq -1 \qquad (1)$$

This yields in particular $\mathrm{x}^6, \mathrm{y}^6, \mathrm{z}^6 \in \mathrm{New}(f_{\mathrm{SOS}})$ and hence $\mathrm{New}(f_{\mathrm{SOS}}) = \mathrm{New}(R_1)$.

Step 3: Write $f_{\mathrm{SOS}} = \sum_{i=1}^{s} g_i^2$ s.t. $g_i \in H_{3,3}$, $s \in \mathbb{N}$. Since $\mathrm{New}(g_i) \subseteq \frac{1}{2}\mathrm{New}(f_{\mathrm{SOS}}) = \frac{1}{2}\mathrm{New}(R_1) = \mathrm{conv}(\mathrm{x}^3, \mathrm{y}^3, \mathrm{z}^3)$, all possible exponents of the g_i's are given by

$$\begin{aligned} &\alpha(1) = (3,0,0)',\ \alpha(2) = (2,1,0)',\ \alpha(3) = (2,0,1)'), \\ &\alpha(4) = (1,2,0)',\ \alpha(5) = (1,1,1)',\ \alpha(6) = (1,0,2)', \\ &\alpha(7) = (0,3,0)',\ \alpha(8) = (0,2,1)',\ \alpha(9) = (0,1,2)^2, \\ &\alpha(10) = (0,0,3)' \end{aligned}$$

and we can write $g_i = \sum_{j=1}^{10} g_{ij} \mathrm{x}^{\alpha(j)}$ $(i = 1, \ldots, s)$, for some $g_{ij} \in \mathbb{R}$.

Step 4: By (1) and the decomposition of $f_{\mathrm{SOS}} = \sum_{i=1}^{s} g_i^2$, we know $-1 \geq (f_{\mathrm{SOS}})_{(4,2,0)'} = \sum_{i=1}^{s} g_{i,2}^2 + \sum_{i=1}^{s} 2 g_{i,1} g_{i,4}$. Hence, using Young's inequality we obtain

$$\begin{aligned} -1 \geq (f_{\mathrm{SOS}})_{(4,2,0)'} &= \sum_{i=1}^{s} g_{i,2}^2 + \sum_{i=1}^{s} 2 g_{i,1} g_{i,4} \\ &\geq \sum_{i=1}^{s} g_{i,2}^2 - \sum_{i=1}^{s} 2\,|g_{i,1}| \cdot |g_{i,4}| \\ &\geq \sum_{i=1}^{s} g_{i,2}^2 - \sum_{i=1}^{s} g_{i,1}^2 - \sum_{i=1}^{s} g_{i,4}^2 \\ &\geq -1 + \sum_{i=1}^{s} g_{i,2}^2 - \sum_{i=1}^{s} g_{i,4}^2, \end{aligned} \qquad (2)$$

where we used that $(f_{\mathrm{SOS}})_{(6,0,0)'} = \sum_{i=1}^{s} g_{i,1}^2 \leq 1$ must hold. Rearranging (2) yields $\sum_{i=1}^{s} g_{i,2}^2 \leq \sum_{i=1}^{s} g_{i,4}^2$.

Analogous argumentation shows that e.g.

$$\begin{aligned} -1 \geq (f_{\mathrm{SOS}})_{(2,4,0)} &= \sum_{i=1}^{s} g_{i,4}^2 + \sum_{i=1}^{s} 2 g_{i,7} g_{i,2} \\ &\geq \cdots \geq -1 + \sum_{i=1}^{s} g_{i,4}^2 - \sum_{i=1}^{s} g_{i,2}^2 \end{aligned}$$

and therefore $\sum_{i=1}^{s} g_{i,2}^2 \geq \sum_{i=1}^{s} g_{i,4}^2$. We finally obtain the equality $\sum_{i=1}^{s} g_{i,2}^2 = \sum_{i=1}^{s} g_{i,4}^2$. Hence, equality must hold everywhere in (2), which shows:

(III) $(f_{\mathrm{SOS}})'_{(6,0,0)} = \sum_{i=1}^{s} g_{i,1}^2 = -1$.
(IV) By Young's Inequality: $g_{i,1} \neq 0$ if and only if $g_{i,4} \neq 0$ and in this case $|g_{i,1}| = |g_{i,4}|$ $(i \in [s])$.
(V) $\mathrm{sign}(g_{i,1}) = -\mathrm{sign}(g_{i,4})$ $(i \in [s])$.
(VI) $(f_{\mathrm{SOS}})_{(4,2,0)'} = -1$.

Clearly, similar observations as in (III)-(V) can be made for all pairs $(g_{i,r}, g_{i,s})$ s.t. $(r,s) \in \{(1,4),(1,6),(7,2),(7,9),(10,3),(10,8)\}$. In addition, as in (VI), we obtain for the other coefficients as in (1):

$$(f_{\mathrm{SOS}})_{(4,2,0)'} = (f_{\mathrm{SOS}})_{(4,0,2)'} = \ldots = -1. \qquad (3)$$

Step 5: By (3), we have

$$(R_1 - f_{\mathrm{SOS}})_{(4,2,0)'} = (R_1 - f_{\mathrm{SOS}})_{(4,0,2)'} = \ldots = 0.$$

Furthermore, (III) for all possible coefficients leads to

$$(R_1 - f_{\mathrm{SOS}})_{(6,0,0)'} = (R_1 - f_{\mathrm{SOS}})_{(0,6,0)'} = (R_1 - f_{\mathrm{SOS}})_{(0,0,6)'} = 0.$$

To sum up, we now have $\mathrm{x}^6,\ \mathrm{y}^6,\ \mathrm{z}^6,\ \mathrm{x}^4\mathrm{y}^2,\ \mathrm{x}^4\mathrm{z}^2, \mathrm{x}^2\mathrm{y}^4,\ \mathrm{x}^2\mathrm{z}^4,\ \mathrm{y}^4\mathrm{z}^2, \mathrm{y}^2\mathrm{z}^4 \notin \mathrm{supp}(R_1 - f_{\mathrm{SOS}})$. However, the form $R_1 - f_{\mathrm{SOS}}$ is SONC and in particular PSD. Hence all vertices in $V(R_1 - f_{\mathrm{SOS}})$ are even. Since $\mathrm{New}(R_1 - f_{\mathrm{SOS}}) \subseteq \mathrm{New}(R_1)$, the only possible lattice point left is $\mathrm{x}^2\mathrm{y}^2\mathrm{z}^2$. Hence, $\mathrm{New}(R_1 - f_{\mathrm{SOS}}) \subseteq \mathrm{conv}(\mathrm{x}^2\mathrm{y}^2\mathrm{z}^2) = \{\mathrm{x}^2\mathrm{y}^2\mathrm{z}^2\}$ must hold and $R_1 - f_{\mathrm{SOS}}$ would be SOS, which is a contradiction.

For the quarternary quartics case, we can argue similarly.

Lemma 4. It holds $P_{4,4} \neq (\Sigma + C)_{4,4}$. More precisely, we have

$$R_2(\mathrm{x},\mathrm{y},\mathrm{z},\mathrm{w}) = \mathrm{x}^2(\mathrm{x}-\mathrm{w})^2 + \mathrm{y}^2(\mathrm{y}-\mathrm{w})^2 + \mathrm{z}^2(\mathrm{z}-\mathrm{w})^2 + 2\mathrm{xyz}(\mathrm{x}+\mathrm{y}+\mathrm{z}-2\mathrm{w}) \in P_{4,4} \backslash (\Sigma + C)_{4,4}.$$

Proof. The proof follows an analogous argumentation as in Lemma 3. By *Hilbert 1888*, we can without loss of generality choose $f_{\mathrm{SOS}} \in \Sigma_{4,4}$ s.t. $R_2 - f_{\mathrm{SOS}} \in C_{4,4}$ is SONC and does not contain any nonnegative circuit polynomial in three variables in its decomposition. It can be deduced that $\mathrm{New}(f_{\mathrm{SOS}}) = \mathrm{New}(R_2)$ must hold.

Further, using Young's inequality for the coefficients of $\mathrm{x}^3\mathrm{w},\ \mathrm{y}^3\mathrm{w},\ \mathrm{z}^3\mathrm{w}$ it can be seen that

$$\mathrm{x}^4,\ \mathrm{x}^2\mathrm{w}^2,\ \mathrm{y}^4,\ \mathrm{y}^2\mathrm{w}^2,\ \mathrm{z}^4,\ \mathrm{z}^2\mathrm{w}^2 \notin \mathrm{supp}(R_2 - f_{\mathrm{SOS}}).$$

Hence, the only even exponents in $\mathrm{supp}(R_2)$ left as possible lattice points for $R_2 - f_{\mathrm{SOS}}$ are $\mathrm{x}^2\mathrm{y}^2,\ \mathrm{x}^2\mathrm{z}^2,\ \mathrm{y}^2\mathrm{z}^2$. However, this means that $R_2 - f_{\mathrm{SOS}}$ is a PSD form in three variables of degree four, which is SOS by *Hilbert 1888*. Hence, R_2 is SOS as well, which is a contradiction. For this reason, R_2 cannot be SOS+SONC.

We are now able to prove Theorem 1.

Proof. [Theorem 1.] "$\Leftarrow$" is clear by *Hilbert 1888*.

"$\Rightarrow$": The Robinson polynomials from Lemma 3 and Lemma 4 are examples of polynomials in $P_{3,6} \backslash (\Sigma + C)_{3,6}$ and $P_{4,4} \backslash (\Sigma + C)_{4,4}$, respectively. Hence, the claim follows directly from Lemma 2.

4. SEPARATING THE SOS+SONC CONE FROM THE SOS AND SONC CONE

In this section, we present a theorem which shows that the SOS+SONC cone is a proper cone extension of both the SOS and the SONC cones for all $(n,2d) \geq (3,4), (n,2d) \neq (3,4)$. This shows that for all nontrivial $(n,2d)$, the SOS+SONC cone is a better inner approximation of the PSD cone than the single SOS and SONC cones.

Theorem 5. For all $(n,2d) \geq (3,4), (n,2d) \neq (3,4)$ it holds

$$(\Sigma + C)_{n,2d} \nsubseteq (\Sigma_{n,2d} \cup C_{n,2d}).$$

Proof. Similarly as in Lemma 2 one can show that it suffices to consider the cases $(n,2d) \in \{(3,6),(4,4)\}$. Hence, the claim follows by constructing explicit examples for the two elementary cases. Indeed, we have e.g.

$$\begin{aligned} f_1 =& \mathrm{x}^4\mathrm{y}^2 + \mathrm{x}^2\mathrm{y}^4 + \mathrm{z}^6 - 3\mathrm{x}^2\mathrm{y}^2\mathrm{z}^2 \\ &+ 1/2 \cdot (\mathrm{z}^3 + 2\mathrm{xyz} + \mathrm{x}^2\mathrm{y})^2 \in (\Sigma + C)_{3,6} \backslash (\Sigma \cup C)_{3,6}, \\ f_2 =& \mathrm{x}^2\mathrm{y}^2 + \mathrm{x}^2\mathrm{z}^2 + \mathrm{y}^2\mathrm{z}^2 + \mathrm{w}^4 - 4\mathrm{wxyz} \\ &+ (\mathrm{xy} + \mathrm{xz} + \mathrm{yz})^2 + \mathrm{w}^4 \in (\Sigma + C)_{4,4} \backslash (\Sigma \cup C)_{4,4}. \end{aligned}$$

5. CONCLUSION

Combining Theorem 1 and Theorem 5 we have shown that for all non Hilbert cases $(n,2d) \geq (3,4),\ (n,2d) \neq (3,4)$, it holds $(\Sigma_{n,2d} \cup C_{n,2d}) \subsetneq (\Sigma + C)_{n,2d} \subsetneq P_{n,2d}$.

We presented explicit examples R_1, R_2 and f_1, f_2 showing the inequalities for the two basic cases and demonstrated how two scale them to arbitrary cases. In terms of polynomial optimization, this shows that for all mentioned cases of n and $2d$, there are PSD polynomial which can be handled by the SOS+SONC cone but neither the SOS nor the SONC cone themselves. On the other hand, there are polynomials which are not classifiable as being PSD by SOS+SONC. It remains to find an efficient way to actually decide membership to the combined SOS+SONC cone.

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