Inner approximations of the PSD cone: What is SOS+SONC?

Universität Konstanz

Moritz Schick (University of Konstanz, Germany, moritz.schick@uni-konstanz.de)

Central question: Given an arbitrary, multivariate, real polynomial $f \in \mathbb{R}[\mathbf{x}_1, \ldots, \mathbf{x}_n]$, is it (globally) nonnegative?

- This question is closely related to polynomial optimization.
- Answering the question is in general co-NP-hard.
- Hence, one is interested in finding sufficient conditions that imply the nonnegativity of a polynomial.

Preliminaries

- $H_{n,2d}$ the vector space of real homogeneous polynomials (forms) in $n \in \mathbb{N}$ variables of degree $2d \in 2\mathbb{N}$.
- $P_{n,2d} := \{f \in H_{n,2d} \mid \forall x \in \mathbb{R}^n : f(x) \geq 0\}$ the cone of positive semidefinite polynomials (PSD).
- $\Sigma_{n,2d} := \left\{ f = \sum_{i=1}^{s} g_i^2 \mid g_1, \ldots, g_s \in H_{n,d} \right\}$ the cone of sums of squares polynomials (SOS).
- $C_{n,2d} := \left\{ f = \sum_{i=1}^{s} g_i \mid g_1, \ldots, g_s \in P_{n,2d} \text{ circuit polynomials} \right\}$ the cone of sums of nonnegative circuit polynomials (SONC).

Fact. 1 (Hilbert 1888) [3]. *It holds* $\Sigma_{n,2d} = P_{n,2d}$ *if and only if* $n = 2$ *or* $2d = 2$ *or* $(n, 2d) = (3, 4)$.

Refer to $n \geq 3, 2d \geq 4, (n, 2d) \neq (3, 4)$ as the non-Hilbert cases.

Fact. 2 [4, Prop. 7.2] and [2, Thm. 3.1]. *It holds:*
(1.) $C_{n,2d} \subseteq \Sigma_{n,2d}$ *if and only if* $n = 2$ *or* $2d = 2$ *or* $(n, 2d) = (3, 4)$.
(2.) $C_{2,2} = \Sigma_{2,2}$ *and* $\Sigma_{n,2} \not\subseteq C_{n,2}$ *for all* $n \geq 3$.
(3.) $\Sigma_{n,2d} \not\subseteq C_{n,2d}$ *for all* $(n, 2d)$ *with* $2d \geq 4$.

The Minkowski sum of the SOS and SONC cones

The SOS+SONC cone

$$(\Sigma + C)_{n,2d} := \Sigma_{n,2d} + C_{n,2d} = \operatorname{conv}(\Sigma_{n,2d}, C_{n,2d})$$

contains all homogeneous polynomials $f = f_1 + f_2 \in H_{n,2d}$ that decompose into a SOS part $f_1 \in \Sigma_{n,2d}$ and a SONC part $f_2 \in C_{n,2d}$.

SOS+SONC is a proper cone inside PSD for all non-Hilbert cases

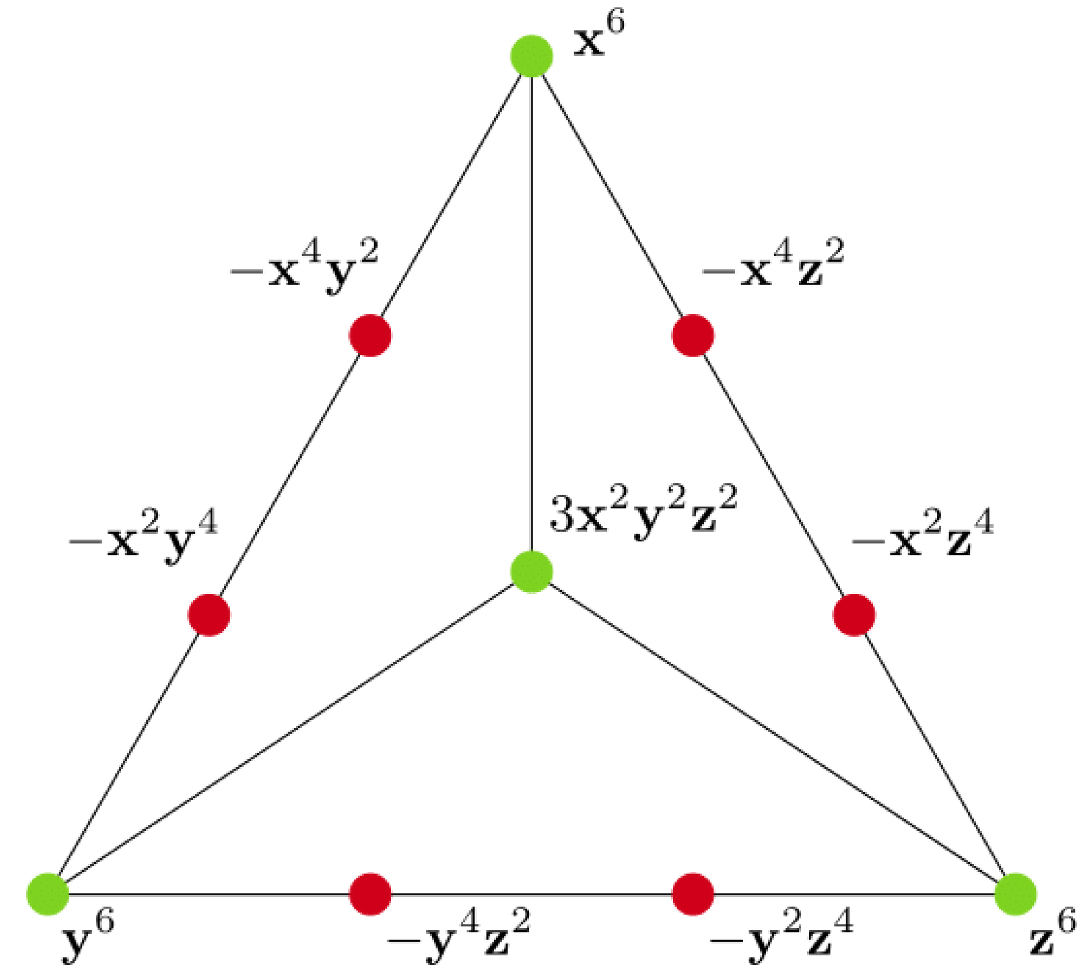
Theorem 1. (Hilbert 1888 analogue for SOS+SONC) It holds $(\Sigma + C)_{n,2d} = P_{n,2d}$ if and only if $n = 2$ or $2d = 2$ or $(n, 2d) = (3, 4)$.

Theorem 1 was proven by Averkov in [1, Cor. 2.17]. Our contribution is to present explicit examples of polynomials in $P_{n,2d} \backslash (\Sigma + C)_{n,2d}$ for all non-Hilbert cases. Such examples could not be found in the literature previously.

Lemma 1. *The Robinson form* $R_1(\mathbf{x}, \mathbf{y}, \mathbf{z}) = \mathbf{x}^6 + \mathbf{y}^6 + \mathbf{z}^6 - (\mathbf{x}^4\mathbf{y}^2 + \mathbf{x}^4\mathbf{z}^2 + \mathbf{y}^4\mathbf{x}^2 + \mathbf{y}^4\mathbf{z}^2 + \mathbf{z}^4\mathbf{x}^2 + \mathbf{z}^4\mathbf{y}^2) + 3\mathbf{x}^2\mathbf{y}^2\mathbf{z}^2 \in H_{3,6}$ *satisfies* $R_1 \in P_{3,6} \backslash (\Sigma + C)_{3,6}$. $(n, 2d) = (4, 4)$*: similar.*

Lemma 2. *If* $f \in P_{n,2d} \backslash (\Sigma + C)_{n,2d}$ *then* $f \in P_{n+1,2d} \backslash (\Sigma + C)_{n+1,2d}$ *and* $\mathbf{x}_1^2 \in P_{n,2(d+1)} \backslash (\Sigma + C)_{n,2(d+1)}$.

Figure 1: Newton polytope of R_1.



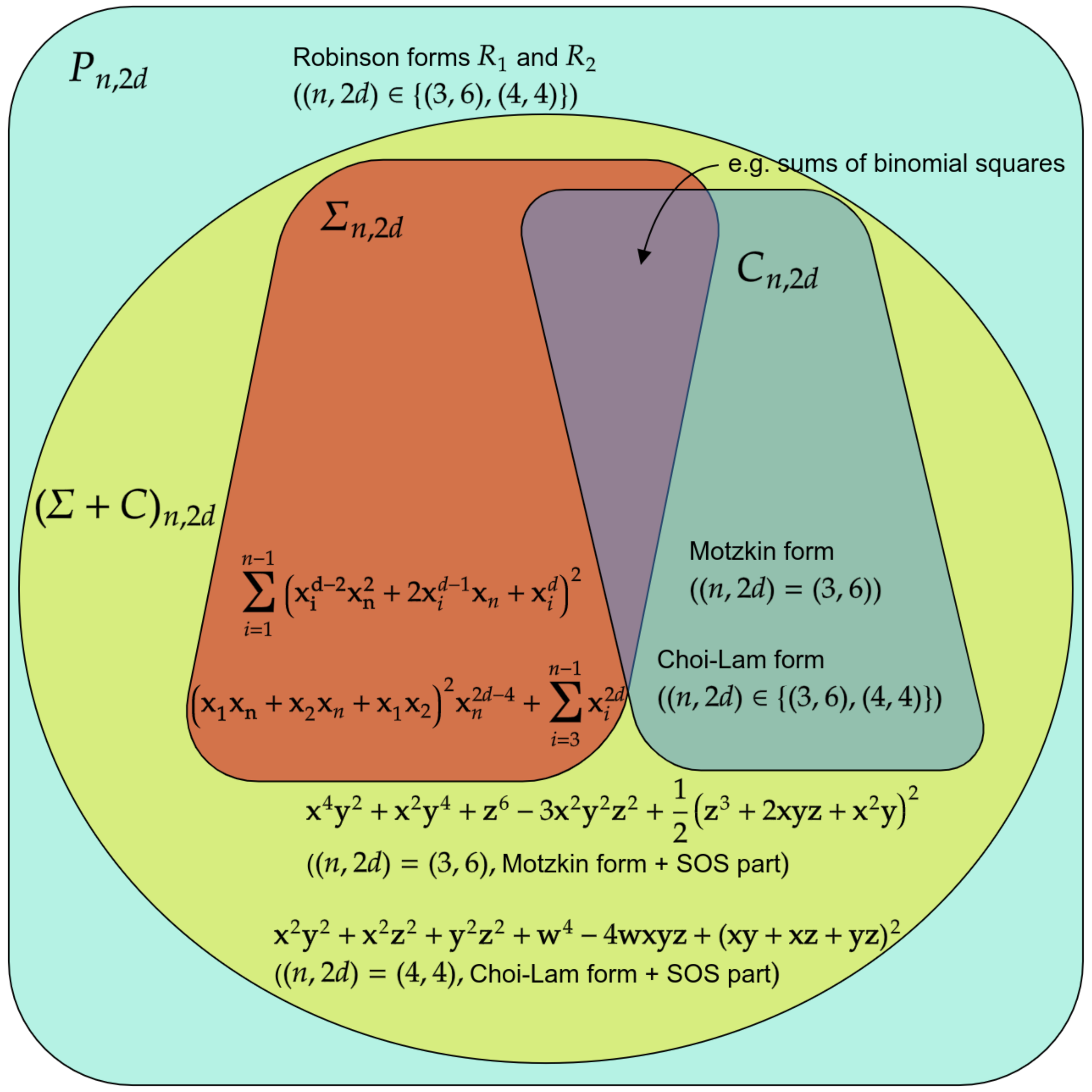
SOS+SONC cone is a nontrivial extension of SOS∪SONC for all non-Hilbert cases

Theorem 2. *It holds* $\Sigma_{n,2d} \cup C_{n,2d} = (\Sigma + C)_{n,2d}$ *if and only if* $n = 2$ *or* $2d = 2$ *or* $(n, 2d) = (3, 4)$.

Open problems / Future work

1. Soon: Combination of SDP and REP approaches for deciding membership to $(\Sigma + C)_{n,2d}$. (GitHub: @schick-moritz)
2. How can SOS+SONC be used in polynomial optimization and how does it compare to SOS/SONC? Can it be used in a hierarchical way like the SOS in Lasserre's hierarchy?
3. Is there a set-theoretic characterization of $\Sigma_{n,2d} \cap C_{n,2d}$?

Figure 2: Graphical illustration of inner approximations of the PSD cone.



References

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