Verifying the independence property of selected ordered fields by defining subrings

Lasse Vogel

Abstract The Shelah conjecture can, when one restricts to only consider ordered fields, be reformulated to "The dependent ordered fields are exactly the almost real closed fields". As it is known that every almost real closed field is indeed dependent, it remains to show that other ordered fields have the independence property. One way to show a given structure has the independence property is to find a definable structure which is independent. One of the standard examples of an independent structure is the ring of integers, what is used to show independence is the existence of infinitely many unrelated primes. Our approach is now to find other rings with many primes definable in fields that we want to verify independence for.

Preliminaries

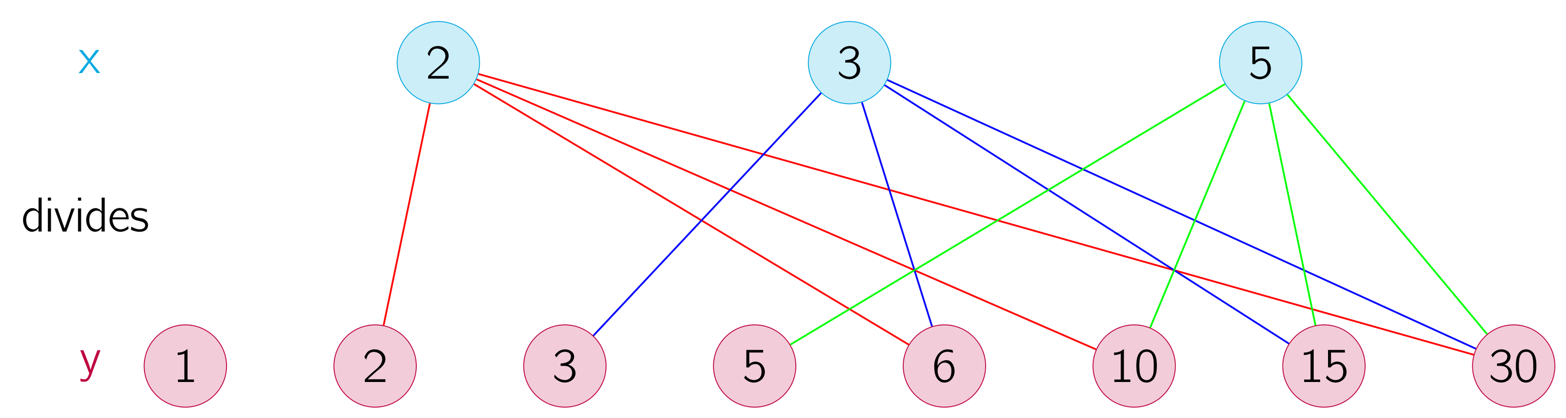
$\mathcal{L}_r := \{+, 0, -, \cdot, 1\}$ — **language of rings**
$\mathcal{L}_{or} := \{+, 0, -, \cdot, 1, <\}$ — **language of ordered rings**

Definition. *An* almost real closed field *is a formally real field K that admits a Henselian valuation v such that the residue field Kv is a real closed field.*

Definition. *Let $\mathcal{L}$ be a language and $\mathcal{M}$ an $\mathcal{L}$-structure. Consider a partitioned $\mathcal{L}$-formula $\varphi(x; y)$ where x is an n-tuple and y an m-tuple of variables. We say that φ* shatters *a set $A \subseteq M^n$ if for each $I \in \mathcal{P}(A)$ there is some $b_I \in M^m$ such that $\varphi(A; b_I) = I$. We say φ has* independence *if there is no finite upper bound for the cardinality of sets that are shattered by φ. As for any $N \in \mathbb{N}$ we can express the existence of an N-sized set which is shattered by φ as a first-order sentence, this property is preserved under elementary equivalence. We say the structure $\mathcal{M}$ is* independent *if some φ has independent over it. Otherwise $\mathcal{M}$ is called* dependent.

Independence of rings with infinitely many primes

As an example consider the language $\mathcal{L} = \mathcal{L}_r$ and take $\mathcal{M} = \mathbb{Z}$. We now want to consider the formula $\varphi(x; y) := x|y = \exists a : x \cdot a = y$. We want to show that this φ has independence. If we choose a finite set of positive primes we can indeed shatter it by choosing the b_I to be the product of all primes in I. For example the set $A = \{2, 3, 5\}$ can be shattered by φ as follows:



As $\mathbb{Z}$ has infinitely many primes, it follows that there is no upper bound for the size of sets that are shattered by φ. Therefore the formula φ has independence.

Note that the only property we required for the shattering was that the set consisted of unrelated primes. As a result, the formula φ has independence in every ring containing infinitely many unrelated primes. With this we obtain for example the independence of polynomial rings (over infinite rings) as the set of polynomials of degree 1 with leading coefficient 1 is an infinite set of unrelated primes.

Shelah conjecture for ordered fields

A conjecture by Shelah suggesting a characterisation of all dependent fields can be specialised to ordered fields as follows:

Conjecture. *(cf. [3] Conjecture 1.2) An ordered field K viewed as structure in the language $\mathcal{L} \in \{\mathcal{L}_r, \mathcal{L}_{or}\}$ is dependent if and only if it is an almost real closed field.*

For every almost real closed field it is known that it is dependent. Hence the open part of the conjecture is to verify that every other real field is independent.

Showing independence with definable subrings

One standard way to show the independence of a field is to find a definable valuation with independent residue field. But this does not work in all required cases, in particular on might only find non-Henselian valuations with real closed residue field. Then the field is not almost real closed but the residue field is clearly dependent (as it is real closed). Instead one can search for definable subrings of the field that contain infinitely many unrelated primes. Then the formula expressing divisibility in this subring would be independent. Therefore the field would be shown to be independent too.

Defining the base field in purely transcendental extensions

We are now in the following setting: Let R be a real closed field and F a purely transcendental extension of R. For every $X \in F \setminus R$ we can now find an ordering where X is infinitesimal with respect to R and another ordering where it is larger than any element of R. As Henselian valuations are convex with respect to every ordering ([1] Lemma 2.1.) both X and X^{-1} must be in the valuation ring. So there is no non-trivial Henselian valuation that is trivial on R. Hence F is not almost real closed.

We now want to show that F is independent. To this end we want to find a definable subring with infinitely many primes. Note that valuation rings do not work as all primes in them are related. Also, while polynomial rings would suffice, it is extremely hard to show whether they are definable or not. Instead we will use the following theorem:

Theorem. *Let R be a real closed field and F a purely transcendental extension of R. Consider the $\mathcal{L}_r$-formulas*

$$\varphi(y) = \exists x : y^2 = x^3 + x$$
$$\psi(x) = \exists y : y^2 = x^3 + x$$

Then $\varphi(F) = R$ and $\psi(F) = R^{\geq 0}$ for the unique ordering on R.

This theorem can be proven by choosing a transcendence basis of F/R indexed by some ordinal and repeatedly applying [2] Lemma 3.2. via transfinite induction. The idea now is to consider F as a function field over R and R as its constants. If we now find a definable preorder we can define the ring of bounded elements.

Exemplary application

(cf. [4] p.39) In the simple case of $F = R(X)$ the rational function field in one variable we can use that the positive semi-definite functions are a definable preorder with defining formula $\rho(x) = \exists a, b : x = a^2 + b^2$. Hence we can define the ring of bounded elements via $\Phi(x) := \exists c : \varphi(c) \wedge \rho(c - x^2)$. The ring defined by Φ are now exactly the pole-free functions with non-positive degree. With some manipulation we can now also get rid of the degree restriction and obtain simply the ring of functions without real poles. But now we are done as the functions $X - c$ for some $c \in R$ are all unrelated primes. This is because by disallowing functions with poles we made it impossible to get rid of roots by multiplication with other functions. Hence we verified that the field $R(X)$ is independent.

Open questions

1. For which fields can we find definable preorders? How do their rings of bounded elements look?
2. Are there different promising ways to define subrings using the definability of the constants?

References

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Contact

Fachbereich Mathematik und Statistik
Universität Konstanz
78457 Konstanz, Germany

lasse.vogel@uni-konstanz.de

Acknowledgement

This work is part of my doctoral research project, which started in 2022 and is supervised by Professor Salma Kuhlmann and mentored by Dr. Lothar Sebastian Krapp at Universität Konstanz.