Fachbereich Mathematik und Statistik
Dr. Vincent Bagayoko
Prof. Dr. Salma Kuhlmann
SoSe 2023

Model Theory

Exercise sheet 1

Structures, formulas, theories, ultrapowers

Throughout the exercise sheet, we fix a first-order language $\mathcal{L}$ over the signature $(K, (f_i)_{i \in I}, (R_j)_{j \in J})$ with corresponding arities $(\alpha_i)_{i \in I}$ and $(\beta_j)_{j \in J}$. We also fix an infinite set X and a free ultrafilter $\mathcal{U}$ on X.

Exercise 1

(4 points)

Show that for any non-empty set M, there is an $\mathcal{L}$-structure $\mathcal{M}$ with underlying set M.

Exercise 2

(4 points)

Among the following formulas in $\mathcal{L}$, which are tautologies?

a) $\forall v_0 \exists v_1 \neg (v_0 = v_1)$

b) $v_0 = v_0$

c) $\exists v_0 \forall v_1 \exists v_2 (f_i(v_0, v_1) = f_i(v_0, v_2))$ (assuming that $\alpha_i = 2$)

d) $\exists v_0 R_j(v_0, c)$. (assuming that $\beta_j = 2$)

Exercise 3

(4 points)

Let $\mathcal{M} = (M, ...)$ be an $\mathcal{L}$-structure where M is finite. Show that the natural embedding $\mathcal{M} \longrightarrow \prod_{\mathcal{U}} \mathcal{M}$ is surjective.

Exercise 4

(4 points)

Let T be a first-order theory over $\mathcal{L}$ and assume that for each $n \in \mathbb{N}$, there is a model of T with cardinality $\geqslant n$. Show that T has an infinite model.

Exercise 5

(4 points)

For each function $f: \mathbb{R} \longrightarrow \mathbb{R}$, we let $\mathcal{R}_f = (R_f, ...)$ denote the ultrapower of $\mathbb{R}_f = (\mathbb{R}, +, \cdot, 0, 1, <, f)$ in the first order language $\langle +, \cdot, 0, 1, <, F \rangle$ where F is a function symbol of arity 1. We write f^* for the interpretation of F in $\mathcal{R}_f$, i.e.

$$f^*([u]_{\mathcal{U}}) = [f \circ u]_{\mathcal{U}}$$

for all $u: X \longrightarrow \mathbb{R}$. We see $\mathbb{R}$ as a subset of R_f according to the natural inclusion.

a) We say that an element $\varepsilon \in R_f$ is infinitesimal if $n \max(\varepsilon, -\varepsilon) < 1$ for all $n \in \mathbb{N}$. Show that there are infinitesimal elements in R_f.

b) Show that f is continuous if and only if for each $r \in \mathbb{R}$ and each infinitesimal $\varepsilon \in R_f$, the quantity $f^*(r+\varepsilon) - f^*(r)$ is infinitesimal.

Please hand in your solutions by ***Friday, 28 April 2023, 10:00*** *(postbox 14 in F4).*