Fachbereich Mathematik und Statistik
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Model Theory

Exercise sheet 2

Ultrapowers, compactness

Throughout the exercise sheet, we fix a first-order language $\mathcal{L}$ over the signature $(K, (f_i)_{i\in I}, (R_j)_{j\in J})$ with corresponding arities $(\alpha_i)_{i\in I}$ and $(\beta_j)_{j\in J}$. We also fix an infinite set X and a free ultrafilter $\mathcal{U}$ on X.

Exercise 4

(4 points)

Let $\mathcal{M}=(M,...)$ be an $\mathcal{L}$-structure where M is finite. Show that the natural embedding $\mathcal{M}\longrightarrow\prod_{\mathcal{U}}\mathcal{M}$ is surjective.

Exercise 5

(6 points)

An oriented graph is an ordered pair (V,E) where V is a non-empty set called the set of vertices and and $E\subseteq V\times V$ is the set of edges. We work in the first-order language $\mathcal{L}_1:=\langle\rightarrow\rangle$ where $\rightarrow$ is a binary relation symbol. So a graph is simply an $\mathcal{L}_1$-structure.

a) A *loop* in a graph (V,E) is an edge $e\in E$ of the form $e=(v,v)$ for some $v\in V$. Is there a first-order theory in $\mathcal{L}_1$ of graphs without loop?

b) A graph is said connected if for all $v,v'\in V$, there are an $n>0$ and $v_1,...,v_n\in V$ with $(v_i, v_{i+1})\in E$ for each $1\leqslant i\leqslant n-1$ and $(v_1,v_n)=(v,v')$. Is there a first-order theory in $\mathcal{L}_1$ of connected graphs?

Exercise 6

(6 points)

For each function $f:\mathbb{R}\longrightarrow\mathbb{R}$, we let $\mathcal{R}_f=(R_f,...)$ denote the ultrapower of $\mathbb{R}_f=(\mathbb{R},+,\cdot,0,1,<, f)$ in the first order language $\langle +,\cdot,0,1,<,F\rangle$ where F is a function symbol of arity 1. We write f^* for the interpretation of F in $\mathcal{R}_f$, i.e.

$$f^*([u]_{\mathcal{U}})=[f\circ u]_{\mathcal{U}}$$

for all $u:X\longrightarrow\mathbb{R}$. We see $\mathbb{R}$ as a subset of R_f according to the natural inclusion.

a) We say that an element $\varepsilon\in R_f$ is infinitesimal if $n\max(\varepsilon,-\varepsilon)<1$ for all $n\in\mathbb{N}$. Show that there are infinitesimal elements in R_f.

b) Show that f is continuous if and only if for each $r\in\mathbb{R}$ each infinitesimal $\varepsilon\in R_f$, the quantity $f^*(r+\varepsilon)-f^*(r)$ is infinitesimal.

*Please hand in your solutions by **Thursday, 11 May 2023, 10:00** (postbox 14 in F4).*