Fachbereich Mathematik und Statistik
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Model Theory

Exercise sheet 3

Categoricity, types, saturation.

Throughout the exercise sheet, we fix a first-order language $\mathcal{L}$ over the signature $(K, (f_i)_{i\in I}, (R_j)_{j\in J})$ with corresponding arities $(\alpha_i)_{i\in I}$ and $(\beta_j)_{j\in J}$.

Exercise 7

(4 points)

Let T be a theory in $\mathcal{L}$. Assume that $|\mathcal{L}| = \aleph_0$ and that there is an infinite cardinal κ such that all models of T of cardinality κ are isomorphic. Show that T is complete.

Exercise 8

(6+2 points)

a) Show that there is a first-order theory T in the language $\langle +, 0\rangle$ of groups whose models are exactly non-trivial, divisible abelian groups without torsion.

b) Show that there is a first-order theory $T_<$ in the language $\langle +, 0, <\rangle$ of ordered groups whose models are exactly non-tirivial, divisible, (totally) ordered abelian groups without torsion.

c) Show that all models of T of same cardinality are isomorphic. Deduce that T is complete.

d) **(bonus)** Show that there are two models of $T_<$ of same cardinality which are not isomorphic.

Exercise 9

(4 points)

Give an example of a complete first-order theory T in a language $\mathcal{L}_0$ of your choice such that there is an $\mathcal{L}_0$-structure $\mathcal{M} = (M, ...)$ and a sequence $(\mathcal{M}_n)_{n\in\mathbb{N}}$ of susbtructures $\mathcal{M}_n = (M_n, ...)$ of $\mathcal{M}$ such that we have $M_n \subseteq M_{n+1}$ and $\mathcal{M}_n \vDash T$ for all $n \in \mathbb{N}$, as well as $M = \bigcup_{n\in\mathbb{N}} M_n$, but

$$\mathcal{M} \nvDash T.$$

Exercise 10

(6 points)

Let $\mathcal{M} = (M, ...)$ be an $\mathcal{L}$-structure and let $A \subseteq M$ be a subset. An n-type $p(v_0, ..., v_{n-1})$ of $\mathcal{M}$ over A is said **complete** if for all formulas $\varphi \in \mathcal{L}_A$ with $\mathrm{FV}(\varphi) \subseteq \{v_0, ..., v_n\}$, we have $\varphi \in p(v_0, ..., v_{n-1})$ or $\neg\varphi \in p(v_0, ..., v_{n-1})$.

a) Show that an n-type of $\mathcal{M}$ over A is complete if and only if it is properpy contained in no n-type of $\mathcal{M}$ over A.

b) Show that every n-type of $\mathcal{M}$ over A is contained in a complete n-type.

c) Show that for $\overline{m} = (m_0, ..., m_n) \in M^n$, the set

$$\mathrm{tp}_A(\overline{m}) := \{\varphi \in \mathcal{L}_A : \mathrm{FV}(\varphi) \subseteq \{v_0, ..., v_n\} \wedge \mathcal{M} \vDash_{\binom{v_0}{m_0} \cdots \binom{v_n}{m_n}} \varphi\}$$

is a complete n-type.

Exercise 11 (bonus)

(6 points)

Let $\mathcal{M} = (M, ...)$ be an $\mathcal{L}$-structure and let κ be an infinite cardinal. Assume that for every subset $A \subseteq M$ with $|A| < \kappa$, each 1-type $p(v_0)$ over A is realised in $\mathcal{M}$. Show that $\mathcal{M}$ is κ-saturated.

Hint: proceed by induction on the length n of a type $q(v_0, ..., v_{n-1})$.

*Please hand in your solutions by **Thursday, 22 May 2023, 10:00** (postbox 14 in F4).*