Fachbereich Mathematik und Statistik
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Model Theory

Exercise sheet 4

Types and definability

Throughout the exercise sheet, we fix a first-order language $\mathcal{L}$ over the signature $(K, (f_i)_{i\in I}, (R_j)_{j\in J})$ with corresponding arities $(\alpha_i)_{i\in I}$ and $(\beta_j)_{j\in J}$.

Exercise 12

(6 points)

Let $\mathcal{N} = (\mathbb{N}_0, +, \cdot, 0, 1, <)$ be the ordered semi-ring of non-negative integers. Let $p \in \mathbb{N}$ be a fixed prime number. Show that the following sets are definable with parameters in $\mathcal{N}$:

a) The set of squares of non-negative integers.

b) The set of prime numbers.

c) The set $\{p^m : m \in \mathbb{N}_0\}$ of powers of p.

Exercise 13

(8 points)

Consider the structure $\mathcal{N}$ of Exercise 12. Let $\mathcal{N}^* = (N^*, \ldots)$ be a *proper* elementary extension of $\mathcal{N}$.

a) Show that every non-empty definable subset of N^* in $\mathcal{N}$ has a minimum. Show that there is a non-empty subset of N^* without a minimum.

b) Show that every infinite definable subset of N^* in $\mathcal{N}^*$ contains an element n^* with $n^* > n$ for all $n \in \mathbb{N}$.

c) Show that the 1-type of $\mathcal{N}$

$$p(v_0) := \{\underbrace{1 + \cdots + 1}_{n \text{ times}} < v_0 : n \in \mathbb{N}\}$$

is contained in two distinct complete 1-types of $\mathcal{N}$ over $\varnothing$.

Exercise 14:

(9+3 points)

Let $\mathcal{M} = (M, \ldots)$ be an $\mathcal{L}$-structure and let $A \subseteq M$ be a subset. We say that $\mathcal{M}$ has *definable choice* over A if for all formulas φ with free variables among $v_0, \ldots, v_n$ and parameters in A, there is a partial function $f: M^n \longrightarrow M$ which is definable over A and such that

$$\mathcal{M} \vDash \varphi(m_0, \ldots, m_{n-1}, f(m_0, \ldots, m_{n-1}))$$

for all $m_0, \ldots, m_{n-1} \in M$ such that $\mathcal{M} \models_{(v_i \mapsto m_i)} \exists v_n \, \varphi(v_0, \ldots, v_{n-1}, v_n)$.

a) Let $\mathcal{M}$ have definable choice over A and let $\mathcal{N} = (N, \ldots)$ be an elementary extension of $\mathcal{M}$. Show that $\mathcal{N}$ has definable choice over A (seen as its image in N).

b) Assume that $\mathcal{M} = (M, +, <, \ldots)$ where $(M, +, <)$ is an Abelian, divisible, totally ordered ordered group, and that $\mathcal{M}$ is o-minimal. Show that $\mathcal{M}$ has definable choice over M.

c) Assume that $\mathcal{N} = (\mathbb{N}, \ldots)$ is an $\mathcal{L}$-structure such that the natural ordering $<$ on $\mathbb{N}$ is definable. Show that $\mathcal{N}$ has definable choice over $\mathbb{N}$.

d) **(bonus)** Let $\mathcal{K} = (K, +, \cdot, 0, 1, \ldots)$ be a strongly minimal structure where $(K, +, \cdot, 0, 1)$ is an infinite field. Show that $\mathcal{K}$ does not have definable choice over K.

Please hand in your solutions by ***Thursday, 6 July 2023, 10:00*** *(postbox 14 in F4).*