Fachbereich Mathematik und Statistik
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Model Theory

Exercise sheet 5

Quantifier elimination

Exercise 15

(6 points)

Let $\mathcal{L} = \langle f \rangle$ where f is a function symbol of arity 1. Let T be the theory in $\mathcal{L}$ containing the following axioms:

A1. $\forall v_0 \exists v_1 (f(v_1) = v_0)$.

A2. $\forall v_0 \forall v_1 ((f(v_0) = f(v_1)) \longrightarrow (v_0 = v_1))$.

a) Show that T is consistent and determine its models.

b) Is T complete?

c) Does T have quantifier elimination in $\mathcal{L}$?

Exercise 16

(6 points)

Let T_{ac} be a theory in the language $\mathcal{L}_r = \langle +, -, \cdot, 0, 1 \rangle$ of rings whose models are exactly algebraically closed fields. Let $k \vDash T_{\mathrm{ac}}$. Let $n \in \mathbb{N}$ and let $I \subseteq k[X_1, \ldots, X_n]$ be a prime ideal. Show (using model theoretic arguments!) that there is an $\overline{a} \in k^n$ such that $Q(\overline{a}) = 0$ for all $Q \in I$.

Exercise 17

(10 points)

Let T_{rc} be a theory in the language $\mathcal{L}_{\mathrm{or}} = \langle +, -, \cdot, 0, 1, < \rangle$ of ordered rings whose models are ordered fields in which monic irreducible polynomials of degree > 1 are of the form $X^2 + b\,X + c$ where $b^2 - 4\,c < 0$.

a) Justify that T is consistent.

b) Show that T has quantifier elimination in $\mathcal{L}_{\mathrm{or}}$.

c) Show that T is o-minimal.

d) Does T have quantifier elimination in $\mathcal{L}_r$?

Exercise 18 (bonus)

(8 points)

Let T_0 be a theory in $\mathcal{L}_r$ whose models are exactly algebraically closed fields of characteristic 0.

Consider a free ultrafilter $\mathcal{U}$ on the set $\mathbb{P}$ of prime natural numbers, and the corresponding ultrapower $\mathcal{F} = \prod_{\mathcal{U}} (\widetilde{\mathbb{F}_p})_{p \in \mathbb{P}}$ where each $\widetilde{\mathbb{F}_p}$ is an algebraic closure of $\mathbb{F}_p$ seen as an $\mathcal{L}_r$-structure.

a) Show that $\mathcal{F}$ is a model of T_0.

b) Assume for contradiction that there are a $\mathcal{C} = (C, \dots) \vDash T_0$, an $n \in \mathbb{N}$ and a polynomial function[1] $C^n \longrightarrow C^n$ which is injective but not surjective.

   i. Show that there are a $p \in \mathbb{P}$ and a polynomial function $(\widetilde{\mathbb{F}_p})^n \longrightarrow (\widetilde{\mathbb{F}_p})^n$ which is injective but not surjective.

   ii. Show that there are a $k \in \mathbb{N}$ and a polynomial function $\mathbb{F}_{p^k}^n \longrightarrow \mathbb{F}_{p^k}^n$ which is injective but not surjective.

   iii. Conclude.

Please hand in your solutions by ***Thursday, 20 July 2023, 10:00*** *(postbox 14 in F4).*

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1. i.e. a function of the form $(a_1, \dots, a_n) \longmapsto (P_1(a_1, \dots, a_n), \dots, P_n(a_1, \dots, a_n))$ where $P_1, \dots, P_n \in C[X_1, \dots, X_n]$.