Fachbereich Mathematik und Statistik
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Real Algebraic Geometry II

Exercise Sheet 1

Valued modules and linear orders

Let Z be a commutative ring with 1. All modules we consider are commuative Z-modules.

Exercise 1

(4 points)

Let $(A, \leq_A)$ be a countable dense linear order without endpoints. Let $(B, \leq_B)$ be an arbitrary countable linear order. Show that $(B, \leq_B)$ is isomorphic to a subordering of $(A, \leq_A)$.

In particular, every countable ordinal embeds into $(\mathbb{Q}, \leq)$.

Exercise 2

(4 points)

Let $v\colon Z[x] \longrightarrow \mathbb{N}_0 \cup \{\infty\}$ be given by $v(0) = \infty$ and $v(p) = \min\{k \in \mathbb{N}_0 : a_k \neq 0\}$ for each $p = \sum_{k=0}^{n} a_k x^k$ in $Z[x] \setminus \{0\}$.

a) Suppose that $Z = \mathbb{Z}$.

  (i). Show that $(Z[x], v)$ is a valued module.

  (ii). Determine the skeleton of $(Z[x], v)$. Hence, or otherwise, find a sum $\coprod_{\gamma \in \Gamma} B(\gamma)$ such that

$$(Z[x], v) \simeq \Big(\coprod_{\gamma \in \Gamma} B(\gamma), v_{\min}\Big).$$

b) Does (i) also hold when Z is an arbitrary commutative ring with 1? Justify your answer.

Exercise 3

(4 points)

Let (M_1, v_1) and (M_2, v_2) be two valued modules with value sets $\Gamma_1 = v_1(M_1 \setminus \{0\})$ and $\Gamma_2 = v_2(M_2 \setminus \{0\})$. Moreover, let $h\colon M_1 \longrightarrow M_2$ be an isomorphism of Z-modules which preserves the valuation.

(i). Let $\tilde{h}\colon \Gamma_1 \longrightarrow \Gamma_2$, $v_1(x) \mapsto v_2(h(x))$. Show that $\tilde{h}$ is well-defined and an isomorphism of ordered sets, i.e. an order-preserving bijection from Γ_1 to Γ_2.

(ii). Show that for each $\gamma \in \Gamma_1$, the map h_γ given by

$$B(M_1, \gamma) \longrightarrow B(M_2, \tilde{h}(\gamma)),\ \pi^{M_1}(\gamma, x) \mapsto \pi^{M_2}(\tilde{h}(\gamma), h(x))$$

is an isomorphism of Z-modules.

Exercise 4
(4 points)

Let $[\Gamma, \{B(\gamma) : \gamma \in \Gamma\}]$ be an ordered system of torsion-free modules.

(i). Show that $\mathbf{H}_{\gamma\in\Gamma} B(\gamma)$ is a module and that $\bigsqcup_{\gamma\in\Gamma} B(\gamma)$ is a submodule of $\mathbf{H}_{\gamma\in\Gamma} B(\gamma)$.

(ii). Show that $S(\bigsqcup_{\gamma\in\Gamma} B(\gamma)) \simeq [\Gamma, \{B(\gamma) : \gamma \in \Gamma\}] \simeq S(\mathbf{H}_{\gamma\in\Gamma} B(\gamma))$.

Exercise 5 (bonus)
(4 points)

Let $(A, \leq)$ be a linear order. Suppose that there exists a countable subset $B \subseteq A$ such that B is dense in $(A, \leq)$, i.e. for any $a, a' \in A$ with $a < a'$, there exists a $b \in B$ with $a \leq b \leq a'$. Let $C \subseteq A$ be a subset which is well-ordered by $\leq$. Show that C is countable.

In particular, any well-ordered subset of $(\mathbb{R}, \leq)$ *is countable.*

Please hand in your solutions by ***Thursday, 27 April 2023, 10:00*** *(postbox 14 in F4).*