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Real Algebraic Geometry II

Exercise Sheet 10

Convex valuations and the Baer-Krull theorem

Exercise 32

(6 points)

Let $G = (\mathbb{Q}^{>0}, \cdot, <)$ be the ordered multiplicative group of positive rationnal numbers. What is the cardinality of the set of orderings on $\mathbb{Q}((G))$ that are $v_{\min}$-compatible?

Solution. There is exactly one ordering on $\mathbb{Q}$, so by the Baer-Krull theorem, the set of $v_{\min}$-compatible orderings on $\mathbb{Q}((G))$ is in bijection with $\{-1, 1\}^{\dim_{\mathbb{F}_2}(G/2G)}$.

The fundamental theorem of arithmetic tells us that every natural number n has a unique decomposition

$$n = \prod_{p \in \mathbb{P}} p^{n_p}$$

where $\mathbb{P}$ is the set of primes and $(n_p)_{p \in \mathbb{P}} \in \mathbb{N}_0^{\mathbb{P}}$ is a family with finite support. Therefore any strictly positive rational number $q = \frac{m}{n}, m, n \in \mathbb{N}$ has a unique decomposition

$$q = \prod_{p \in \mathbb{P}} p^{q_p}$$

where $(q_p)_{p \in \mathbb{P}} = (m_p - n_p)_{p \in \mathbb{P}}$ is a family of integers with finite support. In other words, writing π for the order preserving bijection $\mathbb{N}_0 \longrightarrow \mathbb{P}$, the group G is isomorphic as a group to the additive group of polynomials with integer coefficients $\mathbb{Z}[X]$, *via* the map

$$\mathbb{Q}^{>0} \longrightarrow \mathbb{Z}[X]\,;\, q \mapsto \sum_{n \in \mathbb{N}_0} q_{\pi_n} X^n.$$

We have $2\,G \simeq 2\,\mathbb{Z}[X] = (2\,\mathbb{Z})[X]$ and

$$G / 2\,G \simeq (\mathbb{Z}[X] / (2\,\mathbb{Z})[X]) \simeq \mathbb{F}_2[X].$$

It follows that $\dim_{\mathbb{F}_2}(G / 2\,G) = \aleph_0$, so there are eaxctly $2^{\aleph_0}$ orderings on $\mathbb{G}((G))$.

Exercise 33

(4 points)

Let $(K, \leq)$ be an ordered field with natural valuation v. Show that there is an ordered field extension L / K with exactly two v-compatible orderings for the natural valuation w on L.

Solution. Let G be the ordered divisible closure of $vK^{\times}$ and let k be the real closure of Kv. Recall that the real closure R of K ahs value group G and residue field k under natural valuation. We have an embedding $R \longrightarrow k((G))$ by Kaplansky's theorem. Let $G \times \mathbb{Z}$ be lexicographically ordered with respect to the usual ordering on $\mathbb{Z}$. Let $L = k((G \times \mathbb{Z}))$. We have inclusions $k((G)) \longrightarrow L$ and $K \longrightarrow R$, so L is an ordered field extension of K.

Since G is 2-divisible, we have $2\,(G\times\mathbb{Z})=G\times(2\,\mathbb{Z})$ and $(G\times\mathbb{Z})/2\,(G\times\mathbb{Z})\simeq\mathbb{F}_2$, so $\dim_{\mathbb{F}_2}((G\times\mathbb{Z})/2\,(G\times\mathbb{Z}))=1$, so by the Baer-Krull theorem, since there is a unique ordering on k, there are exactly $|\{-1,1\}^1|=2$ w-compatible orderings on L for the natural valuation w on L.

Exercise 34 [Exercise A, Rag II Final sheet, 2019]
(4 points)

Let $(L,w)/(K,v)$ be an immediate extension of valued fields. Show that any v-compatible ordering on K extends into a w-compatible ordering on L.

Solution. Let I_K (resp. I_L) be the index set of a basis of $(vK^\times)/2\,(vK^\times)$ (resp. $(wL^\times)/2\,(wL^\times)$) and let $(\pi_i)_{i\in I_K}$ (resp. $(\pi_i)_{i\in I_L}$) be corresponding choices of representatives. So we have Baer-Krull correspondences

$$\begin{aligned} f_K\colon \{-1,1\}^{I_K}\times\mathcal{X}(Kv) &\longrightarrow \mathcal{X}_v(K)\\ f_L\colon \{-1,1\}^{I_L}\times\mathcal{X}(Lw) &\longrightarrow \mathcal{X}_w(L)\end{aligned}$$

where $\mathcal{X}_*(\cdot)$ denotes the set of $*$-compatible positive cones on $\cdot$. Now since $(L,w)/(K,v)$ is immediate, we have $Kv=Lw$ and $vK^\times=wL^\times$, so we can also choose I_K,I_L and $(\pi_i)_{i\in I_K}$ and respectively $(\pi_i)_{i\in I_L}$ to be identical. Let $P\in\mathcal{X}_v(K)$, and let $(\eta,\overline{P})\in\{-1,1\}^{I_K}\times\mathcal{X}(Kv)$ with $P=f_K(\eta,\overline{P})$. Write $\tilde{P}:=f_L(\eta,\overline{P})\in\mathcal{X}_w(L)$. For $a\in K\subseteq L$, the signs of a in L or in K are determined by η and $\overline{P}$, so they coincide. In particular, we have $P\subseteq\tilde{P}$, i.e. $\tilde{P}$ extends P.

Exercise 35 (bonus)
(6 points)

Let $(\Gamma,\leq)$ be a linearly ordered set. Show that there is an ordered field whose principal rank is the oder type of $(\Gamma,\leq)$ and which embeds into any ordered field with that principal rank.

Solution. Let $G=\coprod_{\gamma\in\Gamma}\mathbb{Z}$ with its lexicographic ordering $<$ with respect to $(\Gamma,\leq)$ and $(\mathbb{Z},<)$, and corresponding natural valuation v_G. The principal rank of $(G,+,<)$ is the order type of $v_G(G\setminus\{0\})=(\Gamma,\leq)$. For $\gamma\in\Gamma$, we write $\mathbb{1}_\gamma$ for the element of G with $\mathbb{1}_\gamma(\gamma)=1$ and $\mathbb{1}_\gamma(\eta)=0$ if $\eta\in\Gamma\setminus\{\gamma\}$.

Let $\mathbb{K}=\mathbb{Q}((G))$ be the corresponding Hahn field with its ordering and natural valuation v. Define K to be the subfield $\mathbb{Q}((t^{\mathbb{1}_\gamma})_{\gamma\in\Gamma})$ of $\mathbb{K}$ generated by $\{t^{\mathbb{1}_\gamma}:\gamma\in\Gamma\}$. Its principal rank is the principal rank of $v(K^\times)\subseteq G$. Now $v(K^\times)\supseteq\{v(\prod_{\gamma\in\operatorname{supp} g}(t^{\mathbb{1}_\gamma})^{g(\gamma)}):g\in G\}=G$, so $v(K^\times)=G$ and the principal rank of K is the order type of $(\Gamma,\leq)$.

Let F be an ordered field whose principal rank is the order type of $(\Gamma,\leq)$. Write v_F for the natural valuation on F and w for the natural valuation on the value group $v_F F$. So we can pick a representative $x_\gamma\in P_F$ for each $\gamma\in\Gamma$, with $w\circ v_F\colon\{x_\gamma:\gamma\in\Gamma\}\longrightarrow\Gamma$ is an order-preserving bijection. Write $F_0:=\mathbb{Q}((x_\gamma)_{\gamma\in\Gamma})$. Note that F_0 is the field of fractions of the vector $\mathbb{Q}$-space $\mathbb{Q}[(x_\gamma)_{\gamma\in\Gamma}]$ generated by all elements

$$x^g:=\prod_{\gamma\in\Gamma}x_\gamma^{g(\gamma)}$$

for $g\in G$, so $v_F(x^g)=\sum_{\gamma\in\Gamma}g(\gamma)\,v_F(x_\gamma)$ for the natural valuation v_F in F. Since the elements $w(v_F(x_\gamma))$, $\gamma\in\Gamma$ are pairwise distinct and w is a valuation on $v_F F$, the sign of $v_F(x^g)$ is the least $\gamma\in\Gamma$ with $g(\gamma)\neq0$, i.e. the sign of g in G. It follows since v is a convex valuation that the sign of a finite sum $Q=\sum_{g\in G}q_g\,x^g\in\mathbb{Q}[(x_\gamma)_{\gamma\in\Gamma}]$ is the sign of q_{g_0} where $g_0=\min\{g\in G:q_g\neq0\}$.

In particular, the family $(x_\gamma)_{\gamma\in\Gamma}$ is algebraically independent over $\mathbb{Q}$, so there is a unique field isomorphism $K\longrightarrow F_0$ which sends each $t^{\mathbb{1}_\gamma}$ to x_γ for each $\gamma\in\Gamma$. Applying the previous argument to K instead of F for the choice of representatives $(t^{\mathbb{1}_\gamma})_{\gamma\in\Gamma}$, we see that this isomorphism sends the positve cone of K into that of F, so it is an embedding or ordered fields.

*Please hand in your solutions by **Thursday, 6 July 2023, 10:00** (postbox 14 in F4).*