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Real Algebraic Geometry II

Exercise Sheet 12 (Final)

Convex valuations, Hardy fields and exponential fields

Exercise 36

Let $(L,\leq)$ be an ordered field and let $K\subseteq L$ be a dense subfield, i.e. a subfield such that for all $a,b\in L$ with $a<b$, there is a $c\in K$ with $a<c<b$. Consider the natural valuation v on L.

a) Justify that the valuation $v\upharpoonright K$ induced by v on K is equivalent to the natural valuation on K.

b) Show that the extension $(L,v)/(K,v\upharpoonright K)$ is immediate.

c) Give an example of immediate extension of ordered valued fields L_1/K_1 for the natural valuations, such that K_1 is not dense in L_1.

Solution. a) We have $K_v=L_v\cap K$ which is the set of bounded elements in K, so $v\upharpoonright K$ is equivalent to the natural valuation on K (they have the same valuation ring).

b) Let $\gamma=v(a)$ be an Archimedean class of an $a\in L^\times$. We have $\gamma=v(|a|)$, so we may assume that a is strictly positive, whence $a<2a$. By density, we find a $c\in K$ with $a<c<2a$. Then c is in the same Archimedean class as a, so $v(c)=\gamma$.

Let $\bar{b}\in Lv$ be the residue class of a $b\in L_v$. Let ε be in the maximal ideal I_v of L_v. By density, we find a $d\in K$ with $b<d<b+|\varepsilon|$, so $0<b-d<|\varepsilon|$. Recall that I_v is convex and $|\varepsilon|=\max\{-\varepsilon,\varepsilon\}\in I_v$, so $b-d\in I_v$, which means that $\bar{d}=\bar{b}$. Therefore, the extension $(L,v)/(K,v\upharpoonright K)$ is immediate.

c) Consider the ordered field of generalised series $L_1:=\mathbb{Q}((\mathbb{Q}))$ with the ordered abelian group $(\mathbb{Q},+,<)$ as value group. Let K_1 be the subfield of L_1 generated by $\mathbb{Q}$ and $\{t^g:g\in\mathbb{Q}\}$. Then L_1/K_1 is clearly immediate since the residue field of K_1 is equal to the residue field $\mathbb{Q}$ of L_1, and we have $g=v(t^g)\in v(K_1^\times)$ for each $g\in\mathbb{Q}$.

Let $\mathrm{tr}_\omega\colon L_1\longrightarrow L_1$ be the function sending a series $a\in L_1$ to its trunction of length ω, ie. if the ordinal γ has the same order type as $\operatorname{supp} a$ and $\varphi\colon\gamma\longrightarrow\operatorname{supp} a$ is an order preserving bijection, then

$$\mathrm{tr}_\omega(a)=\sum_{\alpha<\gamma\wedge\alpha<\omega} a(\varphi(\alpha))\,t^{\varphi(\alpha)}.$$

Since K_1 is countable, so is $\mathrm{tr}_\omega(K_1)$. So there is a sequence $u\in\{-1,1\}^{\mathbb{N}}$ such that $a:=\sum_{n\in\mathbb{N}}u(n)\,t^{(n-1)/n}$ does not lie in $\mathrm{tr}_\omega(K_1)$. Any series b in the interval $[a-t^1,a+t^1]$ of L_1 must satisfy $\mathrm{tr}_\omega(b)=\mathrm{tr}_\omega(a)$. Indeed otherwise we would have $\min(\operatorname{supp}(b-a))=v(b-a)\leqslant\frac{n-1}{n}<1$ for a certain $n\in\mathbb{N}$. We deduce that $[a-t^1,a+t^1]\cap K_1=\varnothing$, therefore L_1/K_1 is not dense.

Exercise 37

Let $\mathcal{G}$ denote the ring of germs $[f]$ of functions $f\colon(a,+\infty)\longrightarrow\mathbb{R}$. Let H be a Hardy field.

a) Let $f\in P_H$ (i.e. $f\in H$ is positive infinite). Show that $f'>0$.

b) Let $f, g \in H^{\times}$ such that $v(f), v(g) \neq 0$. Show that

$$v(f) < v(g) \Longrightarrow v(f') < v(g').$$

c) Show that the function

$$\begin{aligned} \exp\colon \mathcal{G} &\longrightarrow \mathcal{G} \\ [f] &\longmapsto [\exp \circ f] \end{aligned}$$

is well-defined.

d) Assume that we have $\exp(H) = H^{>}$. Show that $(H, +, \cdot, 0, 1, <, \exp)$ is an ordered exponential field and that exp is compatible with the natural valuation on H.

Solution. a) Recall that f is the germ of an eventually monotonous function, and that the monotonicity of that function is given by the sign of f'. Since f is positive infinite, that sign cannot be negative (since a strictly decreasing or constant function has an upper bound). So $f' > 0$.

b) Note that $\frac{f'}{g'}$ has a limit at $+\infty$ in $\mathbb{R} \cup \{\pm\infty\}$. Now by l'Hospital's rule, $\frac{f}{g}$ has the same limit. Now a quotient $\frac{a}{b}$ for $a, b \in H^{\times}$ has limit $\pm\infty$ if and only if $v(a) < v(b)$, so the result follows.

c) Let f_0, f_1 be functions with $[f_0] = [f_1]$. Given $a \in \mathbb{R}$ such that f_0 and f_1 coincide on $(a, +\infty)$, so do $\exp \circ f_0$ and $\exp \circ f_1$ coincide on $(a, +\infty)$. Therefore $[\exp \circ f_0] = [\exp \circ f_1]$, so $\exp([f_0]) = [\exp \circ f_0]$ is well-defined.

d) For $f = [f_0]$, $g = [g_0] \in \mathcal{G}$, given an $a \in \mathbb{R}$ such that f_0 and g_0 are both defined on $I = (a, +\infty)$, we have we have $\exp(f + g) = [\exp \circ ((f_0 + g_0) \upharpoonright I)] = [(\exp \circ (f_0) \upharpoonright I) \cdot (\exp \circ (g_0) \upharpoonright I)] = [\exp \circ f_0] \cdot [\exp \circ g_0] = \exp(f) \cdot \exp(g)$. Moreover, if $f > 0$, then given a $b \in \mathbb{R}$ such that $f_0(t) > 0$ for all $t \geqslant b$, we have $\exp \circ f_0 > 1$ on $(b, +\infty)$, so $\exp(f) > 1$. Therefore exp is a strictly increasing morphism $(H, +, <) \longrightarrow (H^{>0}, \cdot, <)$. It is surjective by assumption. So $(H, \exp)$ is an ordered exponential field.

Exercise 38

Let $(K, \leq, \exp)$ be a non-Archimedean ordered exponential field and let $a \in P_K$ be positive infinite. Assume that exp satisfies the following growth property:

$$\forall a \in K, \exp(a) \geqslant a + 1. \tag{1}$$

Let $q \in \mathbb{Q}^{>0}$. Show that we have

a) $2\,a > a + q$.

b) $a^2 > q\,a$.

c) $\exp(\log(a)^2) > a^q$.

d) $\exp(a) > \exp(\log(a)^q)$.

Solution. For each $n \in \mathbb{N}$, we will write $\log_n$ (resp. $\exp_n$) for the n-fold iterate of log (resp. exp) on P_K.

a) Since $a \in P_K$, we have $a > \mathbb{Q}$ so $2\,a = a + a > a + q$.

b) Since $a \in P_K$, we have $a^2 > \mathbb{N}\,a$ so $a^2 > q\,a$.

c) Since $\log(a) \in P_K$, applying b) for $\log(a)$ gives $(\log a)^2 > q \log(a)$. As exp is order preserving, we get

$$\exp(\log(a)^2) > \exp(q \log(a)) = a^q.$$

d) By applying (1) to $b := \log_4(a) \in P_K$, we get $\exp(b) \geqslant b+1$, so

$$\exp_2(b) \geqslant \exp(1)\exp(b).$$

We have $\exp(1) \geqslant 2$ by (1), so a) gives $\exp_2(b) \geqslant 2\exp(b) \geqslant \exp(b) + \log(q) = \log_3(a) + \log(q)$. Now

$$\exp_3(b) \geqslant \exp(\log_3(a) + \log(q)) = q\log_2(a),$$

We deduce that $a = \exp_4(b) \geqslant (\log(a))^q$, so $\exp(a) \geqslant \exp(\log(a)^q)$.