Fachbereich Mathematik und Statistik
Prof. Dr. Salma Kuhlmann
Moritz Schick
Vincent Bagayoko
SoSe 2023

# Real Algebraic Geometry II

Final sheet

Convex valuations, Hardy fields and exponential fields

## Exercise 36

Let $(L, \leq)$ be an ordered field and let $K \subseteq L$ be a dense subfield, i.e. a subfield such that for all $a, b \in L$ with $a < b$, there is a $c \in K$ with $a < c < b$. Consider the natural valuation $v$ on $L$.

a) Justify that the valuation $v \upharpoonright K$ induced by $v$ on $K$ is equivalent to the natural valuation on $K$.

b) Show that the extension $(L, v) / (K, v \upharpoonright K)$ is immediate.

c) Give an example of immediate extension of ordered valued fields $L_1 / K_1$ for the natural valuations, such that $K_1$ is not dense in $L_1$.

## Exercise 37

Let $(K, \leq, \exp)$ be a non-Archimedean ordered exponential field and let $a \in P_K$ be positive infinite. Assume that exp satisfies the following growth property:

$$\forall a \in K, \exp(a) \geqslant a + 1. \tag{1}$$

Let $q \in \mathbb{Q}^{>0}$. Show that we have

a) $2a > a + q$.

b) $a^2 > q\,a$.

c) $\exp(\log(a)^2) > a^q$.

d) $\exp(a) > \exp(\log(a)^q)$.

## Exercise 38

Let $\mathcal{G}$ denote the ring of germs $[f]$ of functions $f\colon (a, +\infty) \longrightarrow \mathbb{R}$. Let $H$ be a Hardy field.

a) Let $f \in P_H$ (Ie. $f \in H$ is positive infinite). Show that $f' > 0$.

b) Let $f, g \in H^\times$ such that $v(f), v(g) \neq 0$. Show that

$$v(f) < v(g) \Longrightarrow v(f') < v(g').$$

c) Show that the function

$$\begin{aligned} \exp\colon \mathcal{G} &\longrightarrow \mathcal{G} \\ [f] &\longmapsto [\exp \circ f] \end{aligned}$$

is well-defined.

d) Assume that we have $\exp(H) = H^{>}$. Show that $(H, +, \cdot, 0, 1, <, \exp)$ is an ordered exponential field and that exp is compatible with the natural valuation on $H$.

*This sheet will not be marked but solutions to final sheet will be uploaded on Thursday 20.07.2023. If you have questions, you may come to our office hours.*